MOMENTS AND PRODUCTS OF INERTIA

1

1.1 INTRODUCTION

Rigid body : *A rigid body has invariable size and shape. The distance between any two particles of the rigid body remains always the same.*

In this book we will be dealing with the motion of rigid bodies.

Moment of Inertia of a body about a line : *If m be the mass of an element of a rigid body, r the distance of the element from a given line, then Σmr^2 is called the moment of inertia of the body about this straight line.*

Thus, to determine the moment of inertia of a body of mass M we take an element of the body, multiply it by the square of its perpendicular distance from the given line. The sum of all such quantities is the moment of inertia of the body about the line.

If this sum be denoted by Mk^2, where M is the mass of the body, then k is called the *radius of gyration* of the body about the given line.

Product of Inertia : If (x, y) be the co-ordinates of an element m of the mass referred to two mutually perpendicular lines Ox and Oy, then Σmxy is called the product of inertia of the body with respect to the lines Ox and Oy.

If mutually perpendicular axes Ox, Oy and Oz be taken in the space and (x, y, z) be the co-ordinates of the element m of the body, then the quantities Σmyz, Σmzx and Σmxy are the products of inertia of the body with respect to the pairs of axes, y and z, z and x, x and y, respectively.

1.2 SOME STANDARD CASES OF MOMENT OF INERTIA

(1) Moment of Inertia of a Rod of Length 2a

Case I. *About an axis through an end perpendicular to the rod.*

Let M be the mass of the rod AB; then mass per unit length $= \dfrac{M}{2a}$.

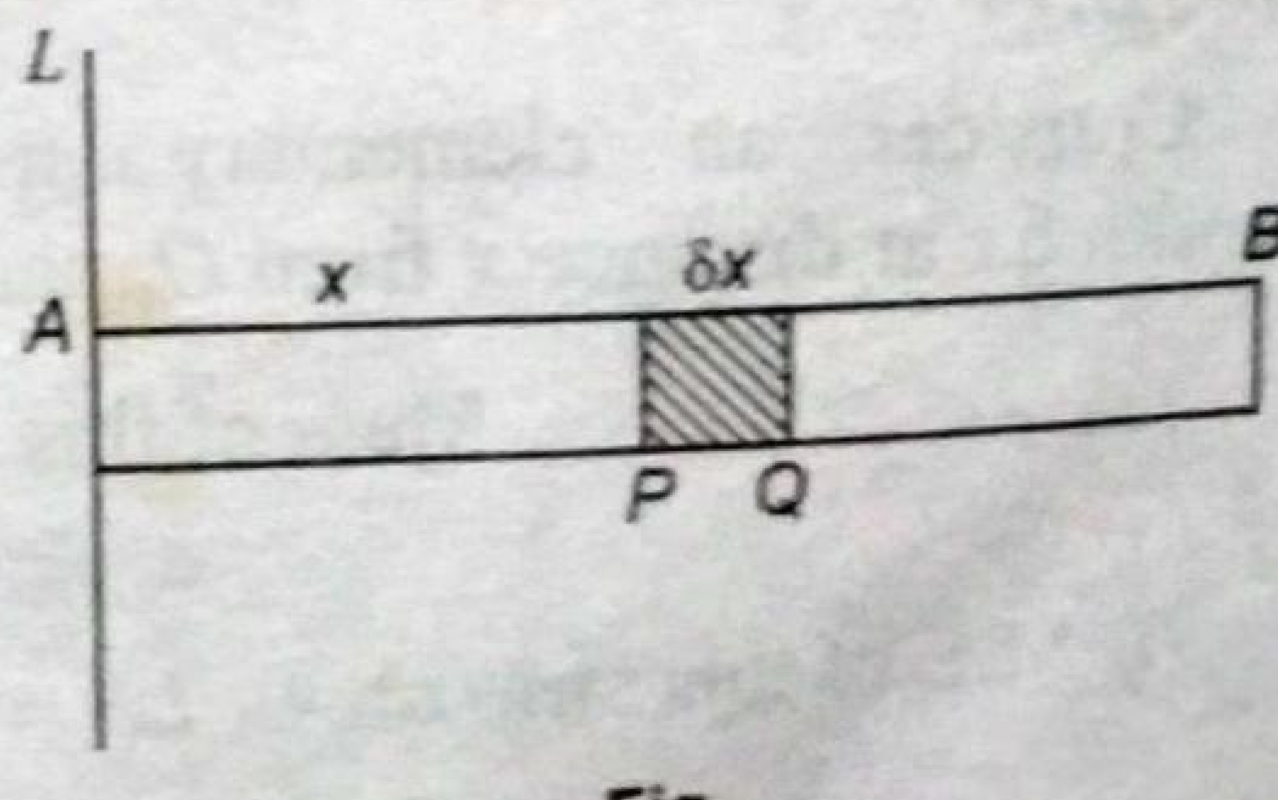

Fig.

Consider an element of breadth δx at a distance x from the end A.

$$\text{Mass of the element} = \frac{M}{2a}\delta x.$$

Its moment of inertia about $AL = \frac{M}{2a}\delta x \cdot x^2$.

Hence, moment of inertia of the rod about $AL = \int_0^{2a} \frac{M}{2a} x^2 dx$

$$= \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = M\frac{4a^2}{3}$$

Case II. *About an axis through the middle point perpendicular to the rod.*

Consider an element PQ of breadth δx at a distance x from the axis LN.

Moment of inertia of this element about LN

$$= \frac{M}{2a}\delta x \cdot x^2.$$

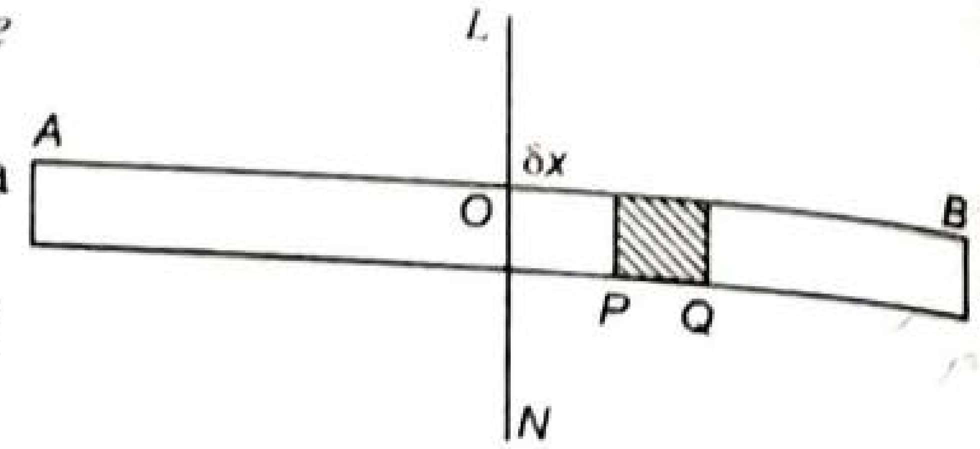

Fig.

$\therefore$ Moment of inertia of the rod AB about LN

$$= \frac{M}{2a}\int_{-a}^{a} x^2 dx$$

$$= \frac{M}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{Ma^2}{3}.$$

(2) Moment of Inertia of a Rectangular Lamina

(*i*) *Moment of inertia of a rectangle about a line through centre parallel to a side.*

Let $ABCD$ be a rectangular lamina such that $AB = 2a$, $AD = 2b$ and centre O. If M be the mass of the lamina, the mass per unit of area $= \frac{M}{4ab}$.

Let OL be an axis parallel to AB through O.

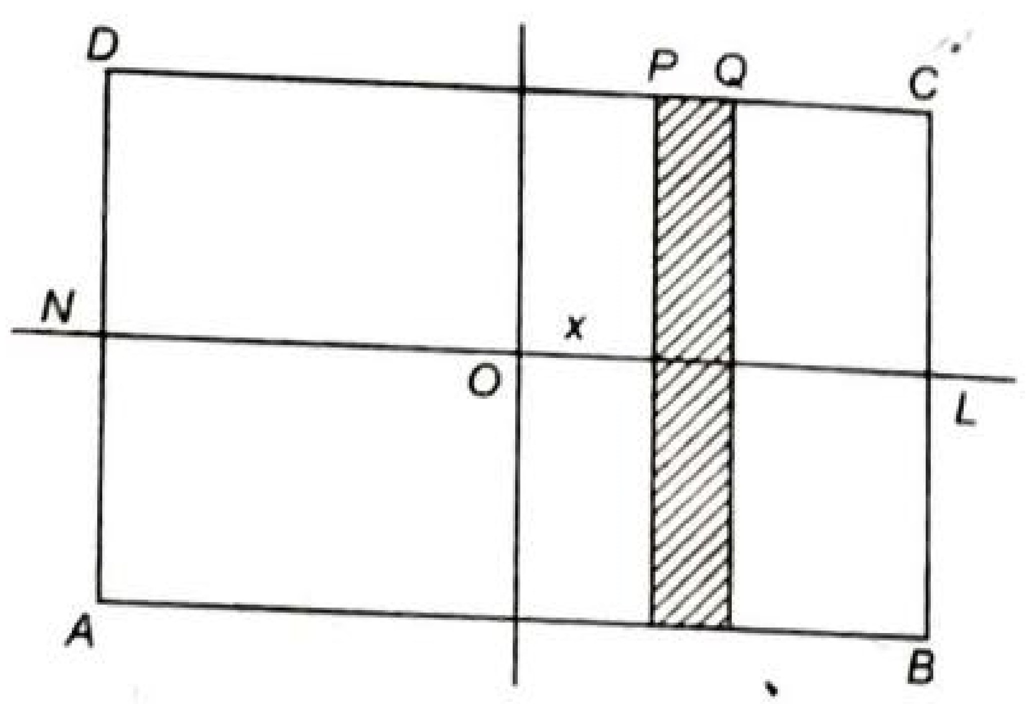

Fig.

Consider an elementary strip of breadth δx at distance x from O, parallel to AD.

$$\text{Mass of this strip} = \frac{M}{4ab}\cdot 2b\delta x = \frac{M}{2a}\delta x.$$

(*i*) M.I. of this strip about $LN = \frac{M}{2a}\delta x\left(\frac{b^2}{3}\right) = \frac{M}{2a}\cdot\frac{b^2}{3}\delta x.$

$\therefore$ M.I. of the rectangle about $LN = \frac{M}{2a}\cdot\frac{b^2}{3}\int_{-a}^{a} dx$

$$= \frac{M}{2a}\cdot\frac{b^2}{3}[x]_{-a}^{a} = \frac{1}{3}Mb^2.$$

Thus, moment of inertia of a rectangular lamina about a line through the centre parallel to the side $2a$ is $\frac{1}{3}Mb^2$.

Similarly moment of inertia of the rectangle about a line through centre parallel to the side $2b$ is $\frac{1}{3}Ma^2$.

(*ii*) *Moment of inertia about a line perpendicular to the lamina and passing through the centre.*

Let OL be a line throught the centre $\perp$ to the lamina.

Consider an elementary area $\delta x\,\delta y$ at a distance $\sqrt{(x^2+y^2)}$ from O.

M.I. of this element about a line OL, perpendicular to the plane $= \rho\,\delta x\,\delta y\,(x^2+y^2)$.

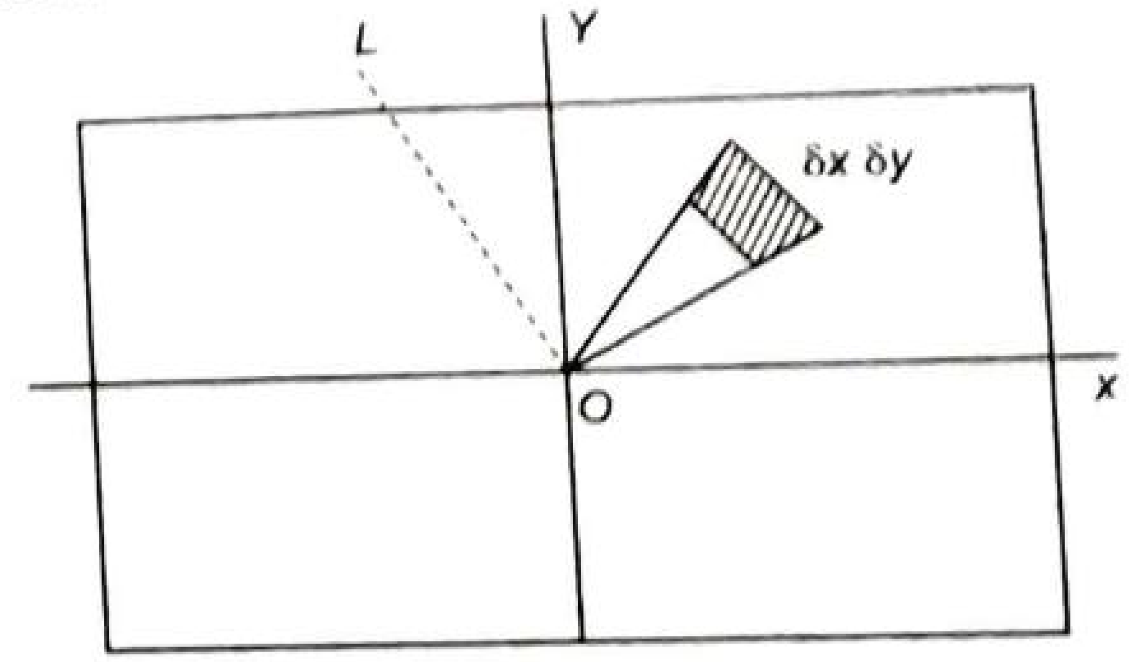

Fig.

$\therefore$ required moment of inertia $= \int_{-b}^{b}\int_{-a}^{a} \rho(x^2+y^2)\cdot dx\,dy$

$$= 4\rho\int_0^b\int_0^a (x^2+y^2)\,dx\,dy$$

$$= 4\rho\left[\frac{x^3}{3}\cdot y + \frac{y^3}{3}x\right]_{x=0,\ y=0}^{x=a,\ y=b}$$

$$= \frac{4\rho}{3}(a^3b + b^3a) = \frac{4ab\rho}{3}(a^2+b^2)$$

$$= \frac{M}{3}(a^2+b^2)$$

as $M = 4ab\rho$.

(3) Moment of Inertia of a Rectangular Parallelopiped

Let $2a, 2b, 2c$ be the lengths of the sides of this parallelopiped, and O its centre. Take OX, OY, OZ axes of the parallelopiped.

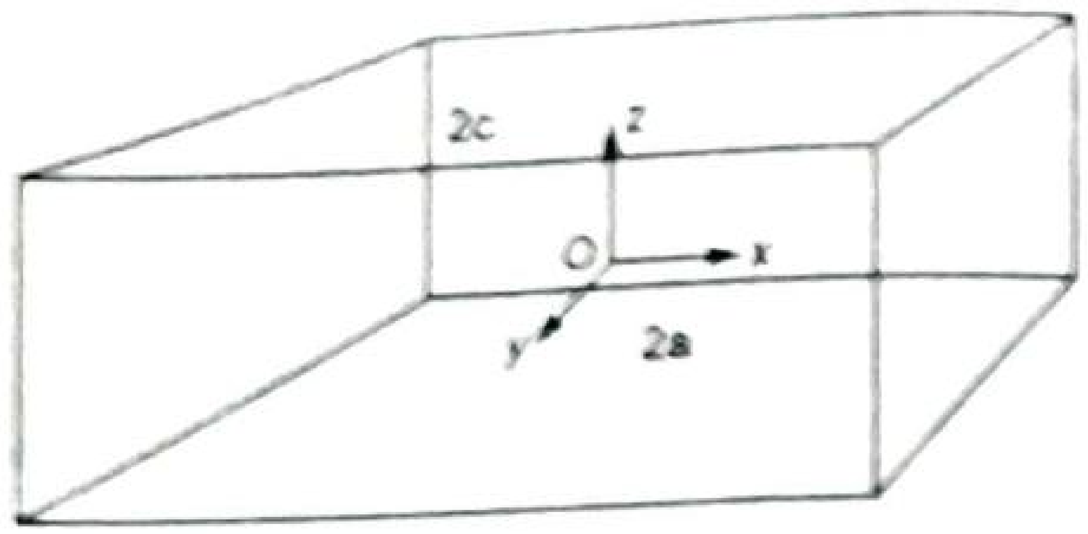

Fig.

Consider an element $\delta x, \delta y, \delta z$ at a point (x, y, z). Moment of inertia of this element about OX (a line parallel to edge $2a$ through O) $= (y^2 + z^2)\rho\, \delta x\, \delta y\, \delta z$.

Hence, the moment of inertia of the whole solid about OX

$$= \int_{-a}^{a} \int_{-b}^{b} \int_{-c}^{c} \rho (y^2 + z^2)\, dx\, dy\, dz$$

$$= \rho \left[\left\{ \frac{y^3}{3} \right\}_{-b}^{b} \{x\}_{-a}^{a} \{z\}_{-c}^{c} + \{x\}_{-a}^{a} \{y\}_{-b}^{b} \left\{ \frac{z^3}{3} \right\}_{-c}^{c} \right]$$

$$= \frac{8\rho\, abc}{3} (b^2 + c^2) = \frac{1}{3} M (b^2 + c^2) \text{ as } M = 8\, abc\, \rho.$$

Thus, moment of inertia of a parallelopiped about a line, through the centre, parallel to the side $2a$ is $\frac{1}{3}\boldsymbol{M(b^2 + c^2)}$.

(4) Moment of Inertia of a Circular Wire

(i) About a diameter

Consider an elementary are $a\,\delta\theta$.

Its mass $= a\,\delta\theta\,\rho$.

Its distance from a diameter $OX = a \sin\theta$.

$\therefore$ Its moment of inertia about $OX = a\delta\theta\rho\,(a\sin\theta)^2$

$= a^2 \rho \sin^2\theta\, \delta\theta$.

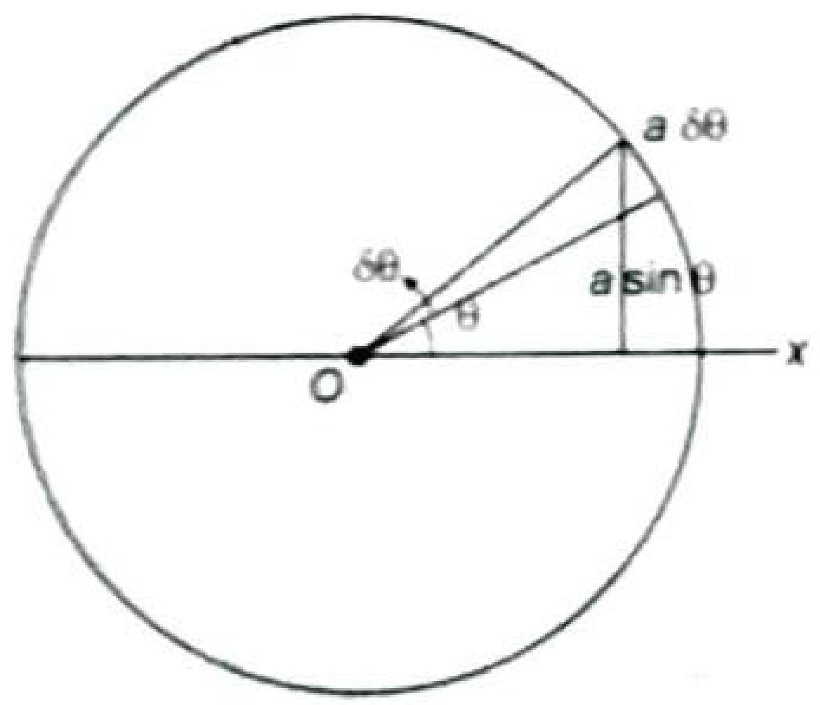

Fig.

Hence, M.I. of the wire about the diameter OX

$$= a^3 \rho \int_0^{2\pi} \sin^2 d\theta$$

$$= 4a^3 \rho \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= 4a^3 \rho \cdot \frac{1}{2} \cdot \frac{\pi}{2} = a^3 \pi \rho = \frac{Ma^2}{2}$$

as $M = 2\pi a \rho$

Thus, moment of inertia of a circular wire about a diameter is $\frac{1}{2}\boldsymbol{Ma^2}$

(ii) About the axis through the centre O

Moment of inertia of an elementary mass about the axis $= \rho a\, d\theta\, a^2$

Hence, M.I. of the circular wire about the axis through O

$$= \int_0^{2\pi} \rho a^3 d\theta = 2\pi a^3 \rho = \boldsymbol{Ma^2}$$

(5) Moment of Inertia of a Circular Plate

(i) To determine moment of inertia of a disc of radius a about its diameter OX, say.

Consider an elementary area $r\,\delta\theta\,\delta r$.

If ρ be the density per unit area, then mass of the element $= r\,\delta\theta\,\delta r\,\rho$, and distance of this element from $OX = r\sin\theta$.

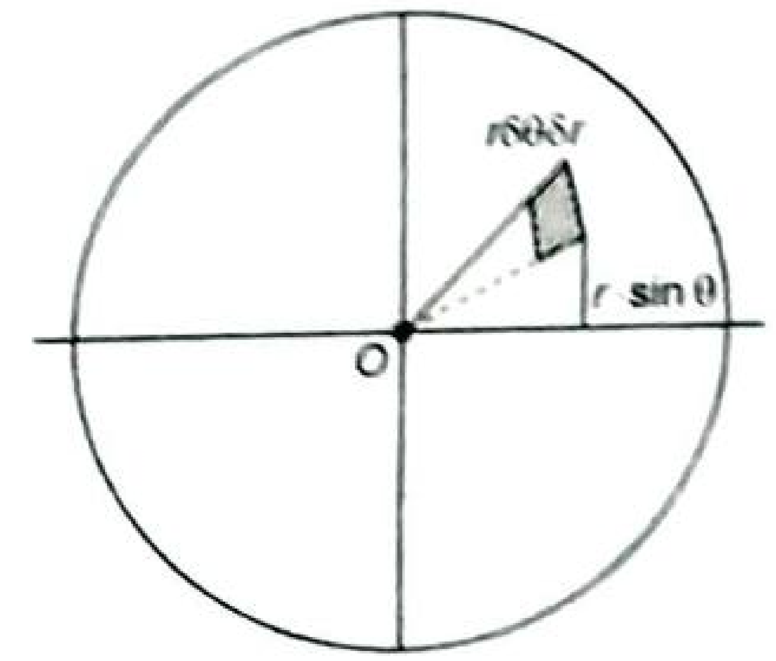

Fig.

$\therefore$ M.I. of the element about OX

$= r\,\delta\theta\,\delta r\,\rho\,(r\sin\theta)^2$

$= r^3 \rho \sin^2\theta\,\delta\theta\,\delta r$,

$\therefore$ M.I. of the disc about the diameter OX

$$= \rho \int_0^{2\pi} \int_0^{a} r^3 \sin^2\theta\, d\theta\, dr$$

$$= 4\rho \int_0^{\pi/2} \int_0^{a} r^3 \sin^2\theta\, d\theta\, dr$$

$$= 4\rho \left[\frac{r^4}{4} \right]_0^a \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2 \rho}{4}$$

$$= \frac{\boldsymbol{Ma^2}}{\boldsymbol{4}} \quad \textbf{as } \boldsymbol{M = \pi a^2 \rho}$$

(ii) To determine the moment of inertia of the disc about a line perpendicular to the disc through the centre O.

Consider the element as above.

Mass of the element $= r\,\delta\theta\,\delta r\,\rho$, and its distance from the axis $= r$.

So its M.I. about the axis $= \rho r\, \delta r\, \delta\theta\, r^2$.

Hence, M.I. of the disc about the axis $= \int_0^{2\pi} \int_0^a \rho r^3 d\theta\, dr$

$$= 4\rho \int_0^{\pi/2} \int_0^a r^3 d\theta\, dr = 4\rho \left[\frac{r^4}{4}\right]_0^a \cdot \frac{\pi}{2}$$

$$= \frac{\pi a^4 \rho}{2} = \frac{\boldsymbol{Ma^2}}{\boldsymbol{2}}$$

Thus, moment of inertia of the disc about the axis is $\frac{1}{2}$ $\boldsymbol{Ma^2}$.

(6) Moment of inertia of an elliptic disc (axes 2a, 2b)

Let us find its *moment of inertia about the major axis OX*.

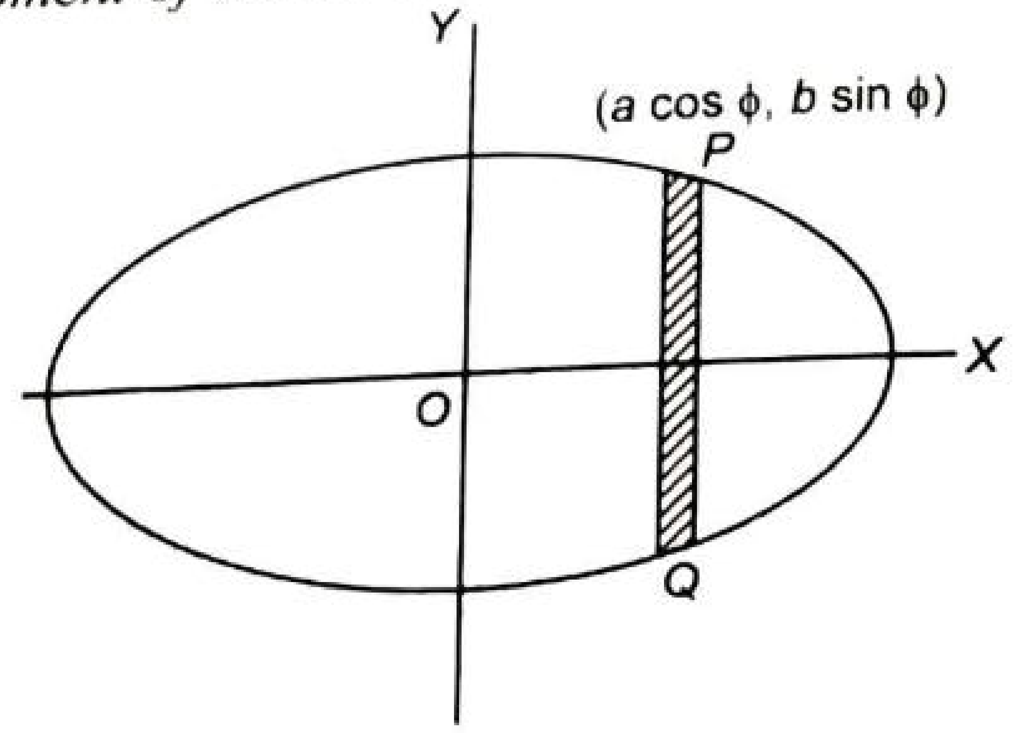

Fig.

Consider an elementary strip PQ, such that P is the point $(a\cos\phi, b\sin\phi)$.

Breadth of the strip $= \delta x = \delta(a\cos\phi) = -a\sin\phi\,\delta\phi$.

Length of the strip $= 2b\sin\phi$.

$\therefore$ mass of the strip $= -2b\sin\phi \cdot \sin\phi\,\delta\phi \cdot \rho$.

Its moment of inertia about $OX = (-2b\sin\phi\; a\sin\phi\,\delta\phi \cdot \rho) \cdot \dfrac{b^2\sin^2\phi}{3}$

$$= -\frac{2}{3}ab^3\rho\sin^4\phi\,\delta\phi.$$

Hence, M.I. of the whole elliptic disc about OX (major axis)

$$= \frac{2}{3}ab^3\rho\int_0^{\pi}\sin^4\phi\,d\phi = \frac{4}{3}ab^3\rho\int_0^{\pi/2}\sin^4\phi\,d\phi$$

$$= \frac{4}{3}ab^3\rho.\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\pi = \frac{1}{4}(\pi ab\rho)\,b^2 = \frac{\boldsymbol{1}}{\boldsymbol{4}}\boldsymbol{Mb^2}$$

as $\quad \boldsymbol{M = \pi ab}\,\rho$.

Thus, moment of inertia of the elliptic disc about the major axis $= \frac{1}{4}\boldsymbol{Mb^2}$.

Similary moment of inertia of the elliptic disc about the minor axis $= \frac{1}{4}\boldsymbol{Ma^2}$

(7) Moment of Inertia of a Hollow Sphere, About a Diameter

The hollow sphere is generated by the revolution of a semi-circular arc about its bounding diameter.

Consider an element $a\,\delta\theta$ of the arc. This when revolved about the diameter AB, generates a circular ring of radius $a\sin\theta$ and width $a\,d\theta$.

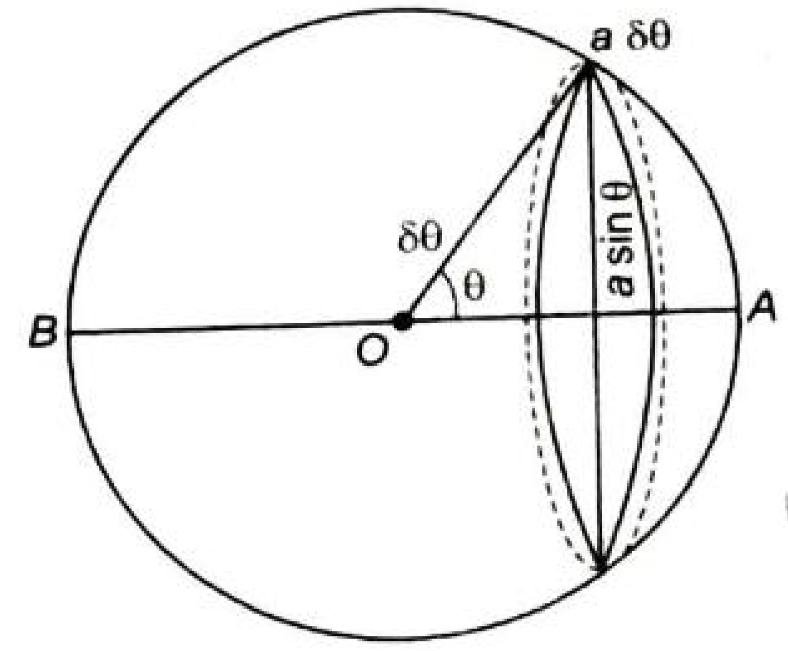

Fig.

Mass of this elementary ring $= 2\pi a\sin\theta\; a\,\delta\theta\rho$.

Distance of every point of this ring from the diameter $AB = a\sin\theta$.

$\therefore$ M.I. of the elementary ring about $AB = (2\pi a\sin\theta\; a\,\delta\theta\,\rho)\,a^2\sin^2\theta$

$$= 2\pi a^4\rho\sin^3\theta\,\delta\theta$$

Hence, M.I. of the hollow sphere about the diameter AB

$$= 2\pi a^4\rho\int_0^{\pi}\sin^3\theta\,d\theta = 4\pi a^4\rho\int_0^{\pi/2}\sin^3\theta\,d\theta$$

$$= 4a^4\rho.\frac{2}{3} = \frac{8\pi a^2\rho}{3} = \frac{2Ma^2}{3}$$

as $\quad M = 4\pi a^2\rho$.

Thus, moment of inertia of a hollow sphere about a diameter is $\frac{2}{3}\boldsymbol{Ma^2}$.

(8) Moment of Inertia of a Solid Sphere, about a Diameter

The solid sphere is generated by the revolution is generated by the revolution of the semicircular area about its bounding diameter.

Consider an elementary area $r\,\delta\theta\,\delta r$ at a distance r from the centre.

This element, when revolved about the diameter AB, generates a circular ring of radius $r\sin\theta$ and cross-section $r\,\delta\theta\,\delta r$.

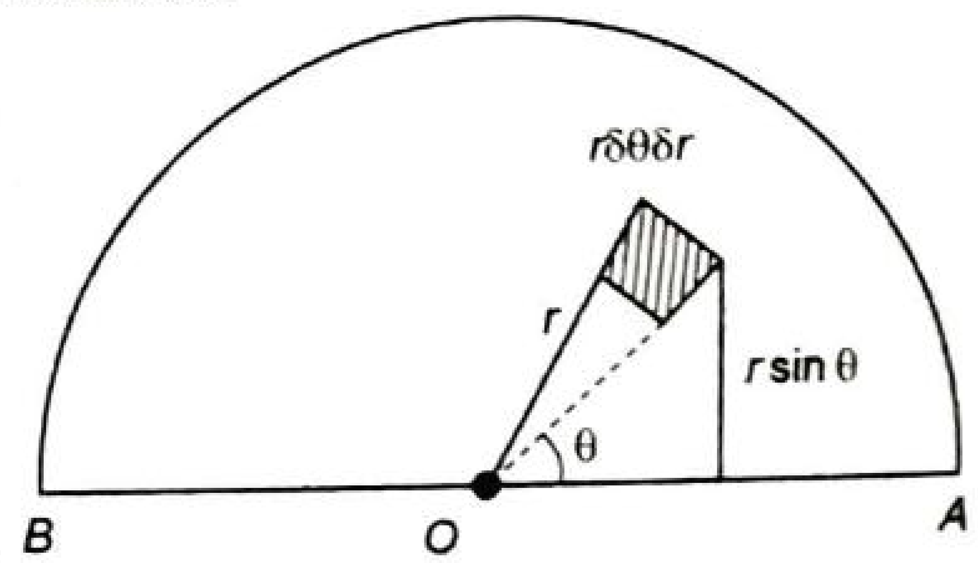

Fig.

Mass of the elementary ring $= 2\pi r\sin\theta \cdot r\,\delta\theta\,\delta\rho \cdot \rho$.

Distance of its every point from $AB = r\sin\theta$

$\therefore$ M.I. of the elementary ring about $AB = (2\pi r \sin\theta \, r\,\delta\theta\,\delta r \cdot \rho)\,(r\sin\theta)^2$

$$= 2\pi\rho\, r^4 \sin^3\theta\,\delta\theta\cdot\delta r$$

Hence, moment of inertia of the sphere about the diameter AB

$$= 2\pi\rho \int_0^a \int_0^\pi r^4 \sin^3\theta\, d\theta\, dr$$

$$= 4\pi\rho \int_0^a \int_0^{\pi/2} r^4 \sin^3\theta\, d\theta\, dr$$

$$= 4\pi\rho \left[\frac{r^5}{5}\right]_0^a \cdot \frac{2}{3} = 4\pi\rho\,\frac{a^5}{5}\cdot\frac{2}{3} = \frac{8\pi\rho a^5}{3.5}$$

$$= \frac{2}{5}\boldsymbol{Ma^2}$$

as $$M = \frac{4}{3}\pi a^3 \rho.$$

Thus, moment of inertia of a solid sphere about a diameter is $\frac{2}{5}\boldsymbol{Ma^2}$.

(9) Moment of Inertia of an Ellipsoid

Let the equation of the ellipsoid be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

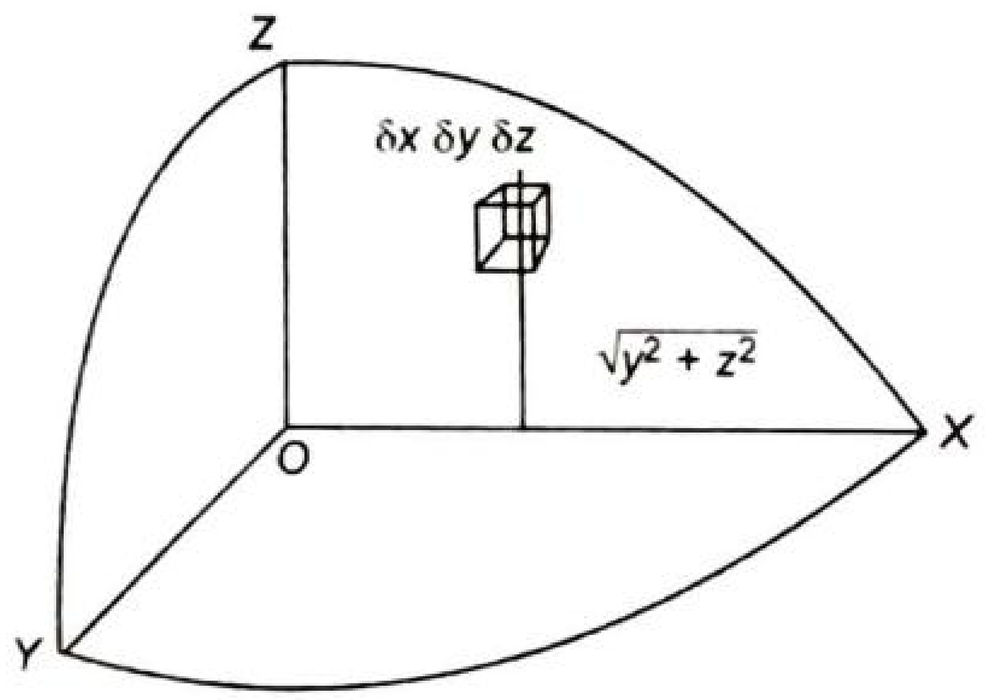

Fig.

Consider an elementary volume $\delta x\,\delta y\,\delta z$ in the positive octant.

Mass of the element $= \rho\,\delta x\,\delta y\,\delta z$

Its distance from $OX = \sqrt{(y^2+z^2)}$

$\therefore$ Its M.I. about $OX = \rho\,\delta x\,\delta z\,(y^2+z^2)$

Hence, moment of inertia of the ellipsoid about OX.

$$= 8\iiint \rho\, dx\, dy\, dz\,(y^2+z^2)$$

where $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\le 1,$

the integration being extended over positive octant of the ellipsoid.

Put $$\frac{x^2}{a^2} = u, \text{ i.e. } x = au^{1/2}$$

So, $$dx = \frac{1}{2}au^{-1/2}du$$

$$\frac{y^2}{b^2} = v, \text{ i.e., } y = bv^{1/2}$$

So, $$dy = \frac{1}{2}bv^{-1/2}dv$$

and $$\frac{z^2}{c^2} = w, \text{ i.e., } z = cw^{1/2},$$

So, $$dz = \frac{1}{2}cw^{-1/2}dw.$$

The total M.I. about $OX = 8\iiint \rho\,\frac{1}{8}abc\,(b^2v + c^2w)\,u^{-1/2}v^{-1/2}w^{-1/2}du\,dv\,dw,$

where $u+v+w\le 1$

$$= abc\rho \iiint \left(b^2 u^{\frac{1}{2}-1} v^{\frac{3}{2}-1} w^{\frac{1}{2}-1} + c^2 u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{3}{2}-1}\right) du\,dv\,dw$$

$$= abc\rho\,(b^2+c^2)\cdot \frac{\Gamma\frac{1}{2}\Gamma\frac{3}{2}\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}} \text{ by Dirichlet's theorem}$$

$$= abc\rho\,(b^2+c^2)\cdot\frac{\pi}{\frac{5}{2}\cdot\frac{3}{2}} = \frac{4abc\rho\pi}{3}\cdot\frac{b^2+c^2}{5}$$

$$= \frac{1}{5}M\,(b^2+c^2)$$

as $$M = \frac{4\pi abc\rho}{3}.$$

Thus, moment of inertia of ellipsoid about the axis 2a is $\frac{1}{5}\boldsymbol{M}\,(\boldsymbol{b^2+c^2})$ *with similar results about the other two axes.*

1.3 REFERENCE TABLE

Below we give standard results obtained above. These results can be divided into three groups as given below :

Group I

Rod of length 2a

about a perpendicular axis through G $\quad M\cdot\frac{1}{3}a^2$

about a perpendicular axis through an end $\quad M\cdot\frac{4}{3}a^2$

Rectangular lamina of sides 2a, 2b

about line through centre parallel to the side $2a$ $\quad M\cdot\frac{1}{3}b^2$

about a perpendicular to its plane through the centre $\quad M\cdot\frac{1}{3}(a^2+b^2)$

Rectangular parallelopiped of edges 2a, 2b, 2c

about a line through its centre parallel to edge $2a$ $\quad M\cdot\frac{1}{3}(b^2+c^2)$

Group II

Circular area of radius a

about a diameter $\quad M\cdot\frac{1}{4}a^2$

about a line ⊥ to the plane through centre $\quad M\cdot\frac{1}{2}a^2$

Elliptic lamina of axes 2a, 2b about the axis $2a$ $\quad M\cdot\frac{1}{4}b^2$

about a perpendicular to its plane through G $\quad M\cdot\frac{1}{4}(a^2+b^2)$

Group III

Sphere of radius a

about a diameter $\quad M\cdot\frac{2}{5}a^2$

Ellipsoid of axes 2a, 2b, 2c about the axis $2a$ $\quad M\cdot\frac{1}{5}(b^2+c^2)$

Routh's rule : Result of all these three groups may be remembered with the help of one Routh's Rule which is as follows

Moment of inertia about an axis of symmetry

$$= \text{mass} \times \frac{\text{sum of squares of perpendicular semi - axes}}{3,\ 4 \text{ or } 5}$$

the denominator is to be 3, 4 or 5 according as the body is rectangular (group I), elliptical (group II) or ellipsoidal (group III).

SOLVED EXAMPLES

EXAMPLE 1 *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being a and b*

Solution Consider a spherical shell of radius x $(a > x > b)$ and of width δx.

If ρ is the density, then moment of inertia of this shell about a diameter

$$= 4\rho\pi x^2 \delta x \cdot \frac{2x^2}{3}$$

Hence, moment of inertia of the given hollow sphere

$$= \int_b^a 4\pi x^2 \rho \cdot dx \cdot \left(\frac{2x^2}{3}\right)$$

$$= \frac{8\pi\rho}{3}\int_b^a x^4\, dx$$

$$= \frac{8\pi\rho}{15}(a^5 - b^5)$$

$$= \frac{2M}{5}\cdot\frac{a^5 - b^5}{a^3 - b^3}$$

as $\quad M = \frac{4}{3}\pi\rho(a^3 - b^3).$

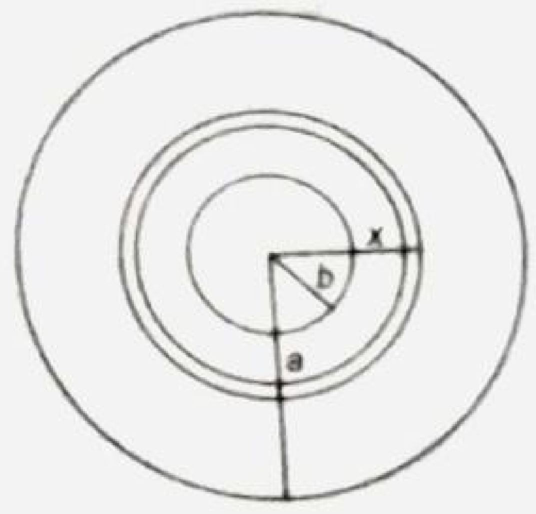

Fig.

EXAMPLE 2 *Show that the moment of inertia of a semi-circular lamina about a tangent parallel to the bounding diameter is*

$$Ma^2\left(\frac{5}{4} - \frac{8}{3\pi}\right)$$

where a is the radius and M is the mass of the lamina

Solution Consider an elementary area $r\,\delta\theta\,\delta r$.

Its distance from the tangent at the vertex, *i.e.* from $AK = a - r\cos\theta$.

Sol its M.I. about $AK = \rho r\,\delta\theta\, dr\,(a - r\cos\theta)^2$

Hence, required moment of inertia

$$= 2\int_0^{\pi/2}\int_0^a \rho r\,\delta\theta\,\delta r\,(a - r\cos\theta)^2$$

$$= 2\rho\int_0^{\pi/2}\int_0^a (a^2 r - 2a r^2\cos\theta + r^3\cos^2\theta)\, d\theta\, dr$$

$$= 2\rho\left\{a^2\left[\frac{r^2}{2}\right]_0^a [\theta]_0^{\pi/2} - 2a\left[\frac{r^3}{3}\right]_0^a [\sin\theta]_0^{\pi/2} + \left[\frac{r^4}{4}\right]_0^a \cdot \frac{1}{2}\cdot\frac{\pi}{2}\right\}$$

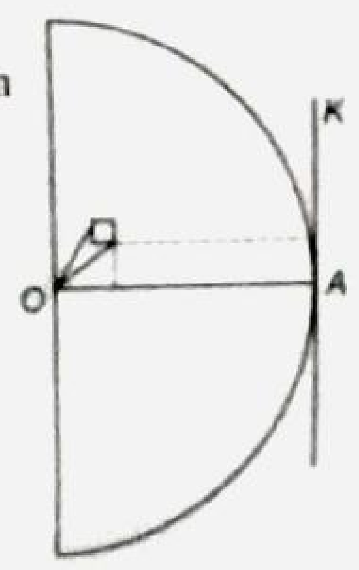

Fig.

$$= 2\rho a^4\left[\frac{\pi}{4} - \frac{2}{3} + \frac{\pi}{1}\right] = 2\rho a^4\left[\frac{\pi}{16} - \frac{2}{3}\right]$$

$$= \frac{1}{2}\pi\rho a^2 . a^2\left[\frac{5}{4} - \frac{8}{3\pi}\right] = Ma^2\left[\frac{5}{4} - \frac{8}{3\pi}\right]$$

as $\quad M = \frac{1}{2}\pi a^2\rho.$

EXAMPLE 3 *Find the moment of inertia of the arc of circle about*

(i) the diameter bisecting the arc.

(ii) an axis through the centre, perpendicular to its plane.

(iii) an axis through its middle point perpendicular to its plane.

Solution Let the arc subtend an angle 2α at the centre O.

Let OA be the diameter bisecting the arc.

Consider an elementary arc $a\,\delta\theta$. Its mass $= \rho a\,\delta\theta$

(i) Its distance from diameter $OA = a\sin\theta$.

Its M.I. about $OA = \rho a\,\delta\theta\,(a\sin\theta)^2$

$= \rho a^3 \sin^2\theta\,\delta\theta,$

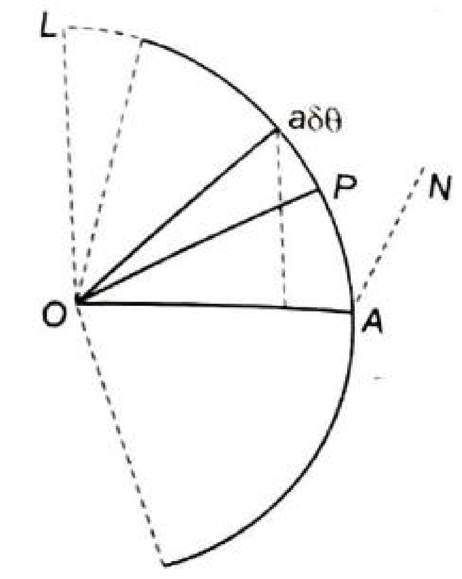

Fig.

Hence, M.I. of the whole arc about

$$OA = \rho a^3 \int_{-\alpha}^{\alpha} \sin^2\theta\, d\theta$$

$$= 2a^3\rho \int_0^{\alpha} \sin^2\theta\, d\theta = a^3\rho \int_0^{\alpha} (1 - \cos 2\theta)\, d\theta$$

$$= a^3\rho\,(\alpha - \sin\alpha\cos\alpha)$$

$$= \frac{Ma^2}{2\alpha}(\alpha - \sin\alpha\cos\alpha)$$

as $\quad M = 2\alpha a\rho.$

(ii) Let OL be a line through centre O perpendicular to the plane of the arc. Distance of $a\,\delta\theta$ from this axis $= a$

M.I. of elementary arc about $OL = (\rho a\,\delta\theta)\,a^2 = \rho a^3\delta\theta$

$\therefore$ M.I. of the whole arc about $OL = \int_{-\alpha}^{\alpha} \rho a^3\, d\theta = 2a^3\alpha\rho$

$= Ma^2$

as $\quad M = 2\alpha a\rho.$

(iii) Again, if AN is the line through the middle point A of the arc perpendicular to the plane of the arc, then distance of $a\,\delta\theta$ from $AN = AP = 2a\sin\frac{1}{2}\theta.$

$\therefore$ M.I. of the elementary arc about $AN = \rho a\,\delta\theta\left(2a\sin\frac{1}{2}\theta\right)^2$

$$= 4a^3\rho\sin^2\frac{1}{2}\theta\,\delta\theta.$$

Hence, M.I. of the whole arc about AN

$$= \int_{-\alpha}^{\alpha} 4a^3\rho\sin^2\frac{1}{2}\theta\, d\theta$$

$$= 4a^3\rho \int_0^{\alpha} (1 - \cos\theta)\, d\theta = 4a^3\rho\,(\alpha - \sin\alpha)$$

$$= \frac{2Ma^2}{\alpha}(\alpha - \sin\alpha).$$

EXAMPLE 4 *Find the product of inertia of a semi-circular wire about diameter and tangent at its extremity.*

Solution OX is the diameter and OY is the tangent at the extremity O.

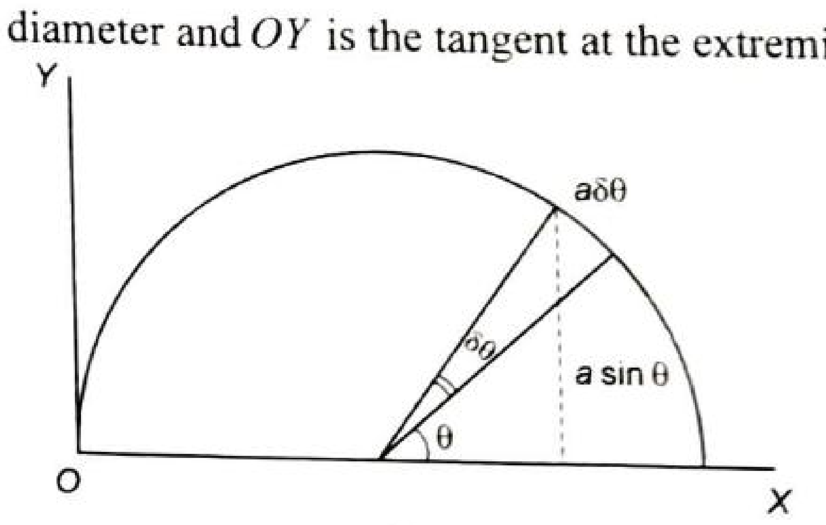

Fig.

Consider the elementary arc $a\,\delta\theta$. Its mass $= \rho a\,\delta\theta$.

Its distance from $OX = a\sin\theta$, and its distance from $OY = a + a\cos\theta$.

$\therefore$ Its product of inertia about OX, OY

$$= a\,\delta\theta\,\rho\,(a\sin\theta)(a + a\cos\theta)$$

$$= \rho a^3 \sin\theta\,(1 + \cos\theta)\,\delta\theta$$

Hence, product of inertia of the wire about OX, OY

$$= \rho a^3 \int_0^{\pi} \sin\theta\,(1 + \cos\theta)\, d\theta$$

$$= \rho a^3\left[-\cos\theta + \frac{1}{2}\sin^2\theta\right]_0^{\pi} = 2\rho a^3.$$

$$= \frac{2Ma^2}{\pi}$$

as $\quad M = \pi a\rho.$

EXAMPLE 5 *Show that the moment of inertia of a paraboloid of revolution about its axis is $\frac{M}{3}\times$ the square of the radius of its base.*

Solution Let the paraboloid of revolution be generated by the revolution of the arc APB of the parabola $y^2 = 4ax$ about the axis AX.

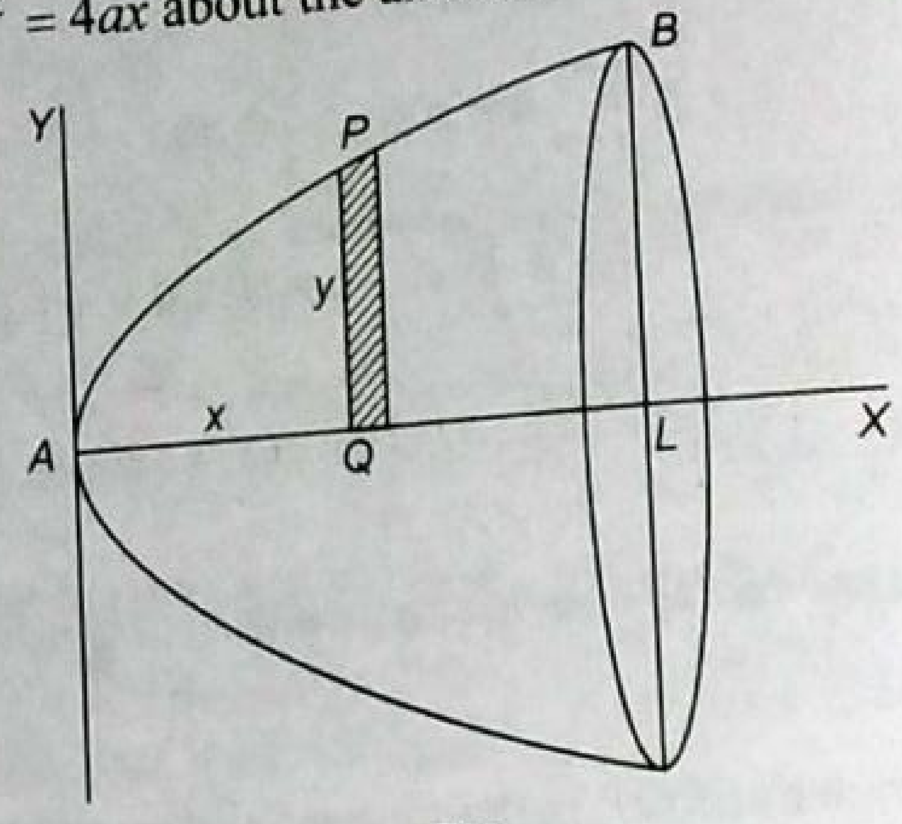

Fig.

Let b be the radius of the base of the paraboloid of revolution,

i.e., y-co-ordinate of $B = b$

$\therefore$ x-co-ordinate of $B = \frac{b^2}{4a}$.

Now consider a strip PQ of breadth δx at a distance x from the vertex. This strip when revolved about x-axis generates a circular disc of radius y.

Mass of the elementary disc so formed $= \pi y^2 \delta x \cdot \rho$. Hence,

M = Mass of the whole paraboloid of revolution

$$= \pi\rho \int_0^{b^2/4a} y^2 dx = 4a\pi\rho \int_0^{b^2/4a} x\, dx$$

$$= 4a\pi\rho \left[\frac{x^2}{2}\right]_0^{b^2/4a} = \frac{\pi\rho b^4}{8a} \quad \ldots(1)$$

M.I. of the elementary disc about the axis $AX = \pi y^2 \delta x \rho \cdot \frac{y^2}{2}$.

Hence, M.I. of the paraboloid of revolution about the axis AX

$$= \frac{\pi}{2}\rho \int_0^{b^2/4a} y^4 dx = \frac{\pi\rho}{2} 16a^2 \int_0^{b^2/4a} x^2 dx$$

$$= \frac{\pi\rho b^6}{24a} = \frac{1}{3}\frac{\pi\rho b^4}{8a} \times b^2 = \frac{1}{3} M \times b^2 \quad \text{from Eq. (1)}$$

$$= \frac{M}{3} \times \text{ the square on the radius of the base.}$$

EXAMPLE 6 *Find the moment of inertia of the area bounded by $r^2 = a^2\cos 2\theta$ about its axis.*

Solution The curve is as shown in the figure.

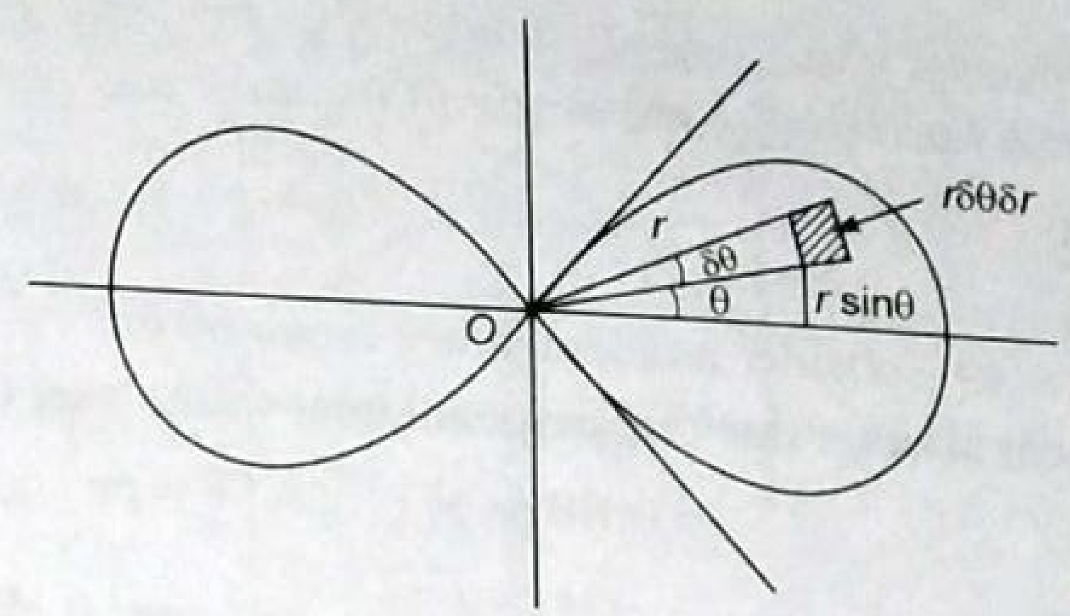

Fig.

The loop is formed between $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$.

Consider an elementary area $r\,\delta\theta\,\delta r$.

Its mass $= \rho \cdot r\,\delta\theta\,\theta r$

Its M.I. about axis $OX = \rho r\,\delta\theta\,\delta r\, r^2 \sin^2\theta = \rho r^3 \sin^2\theta\, d\theta\, dr$.

Hence, M.I. of the whole area (both the loops) about OX

$$= 4\rho \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} r^3 \sin^2\theta\, d\theta\, dr$$

$$= 4p \int_0^{\pi/4} \left[\frac{r^4}{4}\right]_0^{a\sqrt{(\cos 2\theta)}} \sin^2\theta\, d\theta$$

$$= a^4\rho \int_0^{\pi/4} \cos^2 2\theta \cdot \frac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{1}{4}a^4\rho \int_0^{\pi/2} \cos^2 t\,(1-\cos t)\, dt, \quad \text{where } t = 2\theta$$

$$= \frac{1}{4}a^4\rho\left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{2}{3}\right] = \frac{\rho a^4}{16}\left[\pi - \frac{8}{3}\right] \quad \ldots(1)$$

Now if M is the mass of the whole area (for both the loops), then

$$M = 4\int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr$$

$$= \frac{4\rho a^2}{2} \int_0^{\pi/4} \cos 2\theta\, d\theta$$

$$= \rho a^2 \int_0^{\pi/2} \cos t\, dt = \rho a^2$$

Putting this value of M in Eq. (1), required moment of inertia

$$= \frac{Ma^2}{16}\left[\pi - \frac{8}{3}\right].$$

EXAMPLE 7 *Show that the moment of inertia of the area of the lemniscate $r^2 = a^2 \cos 2\theta$ about a line through the origin in its plane and perpendicular to its axis is $\frac{Ma^2(3\pi+8)}{48}$.*

Solution (See fig. ex. 6) Consider an elementary area $r\,\delta\theta\,\delta r$.

M.I. of this element about a line OY perpendicular to the axis OX

$$= (r\,\delta\theta\,\delta r\,\rho)\, r^2 \cos^2\theta = \rho r^3 \cos^2\theta\,\delta\theta\,\delta r.$$

$\therefore$ Required moment of inertia $= 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \cos^2\theta\, d\theta\, dr$

$$= 4\rho \int_0^{\pi/4} \frac{a^4 \cos^2 2\theta}{4} \cos^2\theta\, d\theta$$

$$= \frac{a^4\rho}{4}\int_0^{\pi/2} \frac{1}{4}\cos^2 t\,(1+\cos t)\, dt$$

$$= \frac{\rho a^4}{16}\left(\pi + \frac{8}{3}\right) = \frac{\rho a^4}{48}(3\pi+8)$$

$$= \frac{Ma^2}{48}(3\pi+8)$$

as $M = \rho a^2$ as in the last example.

EXAMPLE 8 *Show that the moment of inertia of the area of the lemniscate $r^2 = a^2\cos 2\theta$ about a line through the origin and perpendicular to the plane is $\frac{1}{8}M\pi a^2$.*

Solution (see fig. ex. 6) Distance of the element $r\,\delta\theta\,\delta r$ from the line $= r$.

$\therefore$ Required moment of inertia $= 4\int_0^{\pi/4}\int_0^{a\sqrt{(a\cos 2\theta)}} (\rho r\, d\theta\, dr)\, r^2$

$$= 4\rho\int_0^{\pi/4} \frac{a^4\cos^2 2\theta}{4}\,\theta$$

$$= \frac{a^4}{2}\rho \int_0^{\pi/2}\cos^2\phi\, d\phi \ldots [2\theta = \phi]$$

$$= \frac{\rho a^4}{2}\cdot\frac{\pi}{4} = \frac{1}{8}M\pi a^2$$

as $M = \rho a^2$.

EXAMPLE 9 *Show that the moment of inertia of parabolic area (of latus rectum 4a) cut off by an ordinate at distance h from the vertex is $\frac{3}{7}Mh^2$ about the tangent at the vertex and $\frac{4}{5}Mah$ about the axis.*

Solution Let the equation of the bounding parabola be $y^2 = 4ax$.

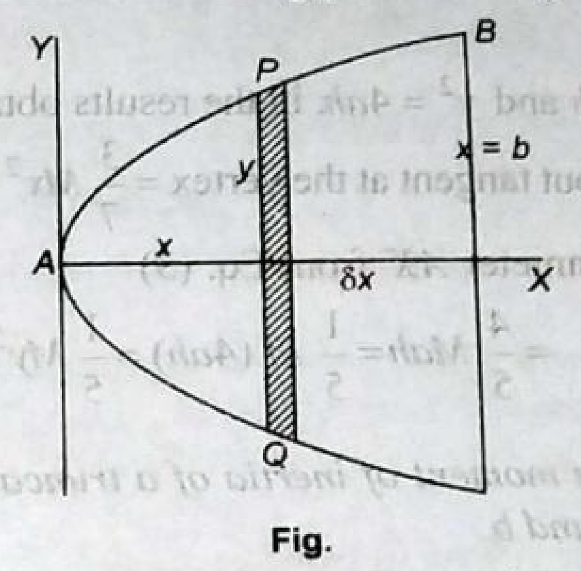

Fig.

Consider strip of breadth δx at a distance x from the vertex.

Mass of the whole parabolic area is

$$M = \int_0^h 2y\rho\, dx = 2\rho\int_0^h \sqrt{(4ax)}\, dx$$

$$= 4\rho\cdot\frac{2}{3}h^{3/2}\sqrt{a} = \frac{8}{3}\rho a^{1/2}h^{3/2}. \quad \ldots(1)$$

Now every point of this strip is at a distance x from the axis of y (tangent at the vertex), hence moment of inertia of this strip about $AY = (2y\,\delta x\,\rho)\cdot x^2 = 2y\rho x^2\delta x$.

$\therefore$ M.I. of the whole area about AY

$$= \int_0^h 2y\rho\, x^2 dx = 2\rho\int_0^h \sqrt{(4ax)}\, x^2 dx$$

$$= 4\rho a^{1/2}\int_0^h x^{5/2}dx = 4\rho a^{1/2}\cdot\frac{2}{7}\cdot h^{7/2} = \frac{8}{7}\rho a^{1/2}h^{7/2}$$

$$= \frac{3M^2}{7} \quad \text{from Eq. (1)}$$

This proves the first result.

Again M.I. of the elementary strip about the axis $AX = (2y\,\delta x\,\rho)\cdot\frac{y^2}{3}$

$\therefore$ M.I. of the whole area about the axis AX

$$= \frac{2}{3}\rho\int_6^h y^3\, dx = \frac{2}{4}\rho\int_6^h (4ax)^{3/2}dx$$

$$= \frac{16}{3}\rho a^{3/2}\int_0^h x^{3/2}dx = \frac{32}{15}\rho a^{3/2}h^{5/2}$$

$$= \frac{4}{5}Mah \quad \ldots(3)$$

This proves the second result.

EXAMPLE 10 *Show that the moment of inertia of the part of the area of parabola cut off by any ordinate at a distance x from the vertex is $\frac{3}{7}Mx^2$ about the tangent at the vertex, and $\frac{1}{5}My^2$ about the principal diameter where y is the ordinate corresponding to x.*

Solution Putting x for h and $y^2 = 4ah$, in the results obtained in the last example, we get moment of inertia about tangent at the vertex $= \frac{3}{7}Mx^2$ from Eq. (2), and moment of inertia about principal diameter AX from Eq. (3)

$$= \frac{4}{5}Mah = \frac{1}{5}M(4ah) = \frac{1}{5}My^2.$$

EXAMPLE 11 *Find the moment of inertia of a truncated cone abont its axis, the radii of its ends being a and b.*

Solution Let $ABCD$ be the truncated cone or frustum of the cone of semi-vertical angle α.

Consider a circular disc of breadth δx at a distance x from the vertex V.

Then radius of disc $= x\tan\alpha$ and its mass

$$= \pi x^2 \tan^2\alpha\ \delta x \rho.$$

$\therefore$ Moment of inertia of this disc about the axis OO'

$$= (\pi x^2 \tan^2\alpha\ \delta x \rho)\cdot\frac{x^2\tan^2\alpha}{2}$$

$$= \frac{1}{2}\pi\rho\tan^4\alpha\cdot x^4\delta x.$$

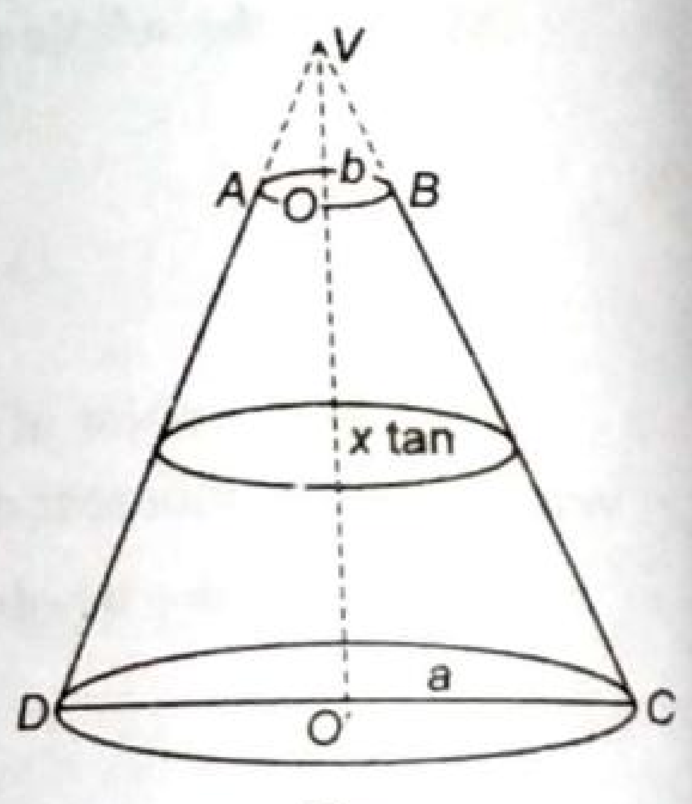

Fig.

Hence, moment of inertia of whole frustum about OO'

$$= \frac{1}{2}\pi\rho\tan^4\alpha\int_{b\cot\alpha}^{a\cot\alpha} x^4 dx$$

$$= \frac{\pi\rho\tan^4\alpha}{2}\cdot\frac{1}{5}(a^5 - b^5)\cot^5\alpha$$

$$= \frac{\pi\rho\cot\alpha}{10}(a^5 - b^5) \quad \text{...(1)}$$

Also mass $M = \int_{b\cot\alpha}^{a\cot\alpha} \pi x^2\tan^2\alpha\ \rho\, dx$

$$= \pi\rho\tan^2\alpha.\frac{\cot^2\alpha}{3}(a^3 - b^3)$$

$$= \frac{\pi\rho\cot\alpha}{3}(a^3 - b^3)$$

or $$\pi\rho\cot\alpha = \frac{3M}{a^3 - b^3}.$$

Putting this value of $\pi\rho\cot\alpha$ in Eq. (1), the required moment of inertia

$$= \frac{3M}{10}\cdot\left(\frac{a^5 - b^5}{a^3 - b^3}\right)$$

EXAMPLE 12 *Show that the moment of inertia of a cone of mass M is $\frac{3}{10}Ma^2$ about its axis, a being the radius of the base.*

Hint Proceed as above or put $b = 0$ in the result of the above example.

EXAMPLE 13 *From a uniform sphere of radius a, spherical sector of vertical angle 2α is removed. Show that the moment of inertia of the remainder of mass M about the axis of symmetry is*

$$\frac{1}{5}Ma^2(1 + \cos\alpha)(2 - \cos\alpha).$$

Solution Let $OABC$ be the spherical sector that has been removed.

M = mass of the remainder

= mass of the sphere – mass of the sector.

$$= \frac{4}{3}\pi a^2\rho - \int_0^\alpha\int_0^a \rho(2\pi r\sin\theta)\, r d\theta\, dr$$

$$= \frac{4}{3}\pi a^3\rho - \frac{2}{3}a^3\pi\rho(1 - \cos\alpha)$$

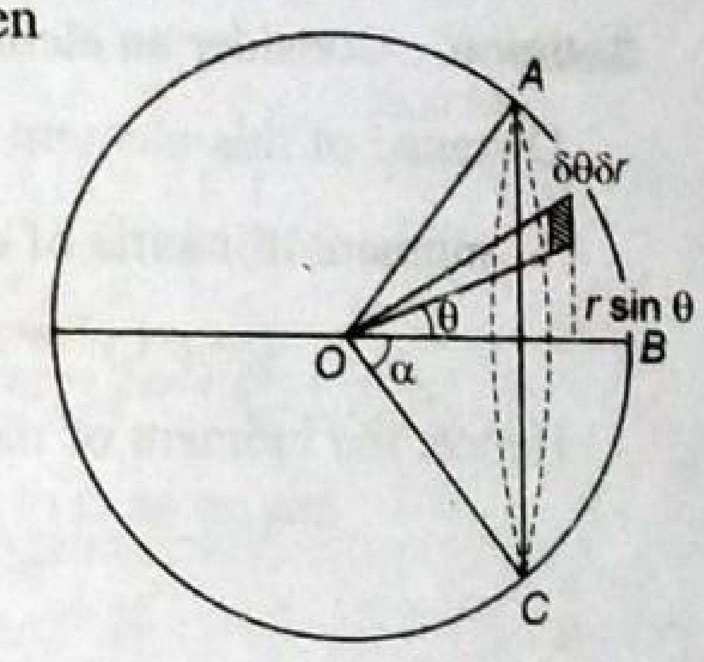

Fig.

i.e., $$M = \frac{2\pi a^3\rho}{3}(1 + \cos\alpha)$$

So that $$\frac{2\pi a^3\rho}{3} = \frac{M}{1 + \cos\alpha} \quad \text{...(1)}$$

Now moment of inertia of the remainder about OB, the axis of symmetry

= M.I. of the sphere – M.I. of the sector

$$= \frac{2}{5}\left(\frac{4}{3}\pi a^3\rho\right)a^2 - \int_0^\alpha\int_0^a \rho(2\pi r\sin\theta)\cdot r^2\sin^2\theta\, r\, d\theta\, dr$$

$$= \frac{8\pi a^5\rho}{15} - \frac{2\pi\rho a^5}{5}\int_0^\alpha\left(\frac{3\sin\theta - \sin 3\theta}{4}\right)d\theta,$$

$$= \frac{8\pi a^5\rho}{15} - \frac{\pi a^5\rho}{10}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right]_0^\alpha$$

$$= \frac{\pi a^5 \rho}{30}[16 - 8 + 9\cos\alpha - \cos 3\alpha]$$

$$= \frac{\pi a^5 \rho}{30}[8 + 9\cos\alpha - (4\cos^3\alpha - 3\cos\alpha)]$$

$$= \frac{2\pi a^5 \rho}{15}[2 + 3\cos\alpha - \cos^3\alpha]$$

$$= \frac{1}{5}\cdot\frac{Ma^2}{(1+\cos\alpha)}(1+\cos\alpha)^2(2-\cos\alpha) \quad \text{[from Eq. (1)]}$$

$$= \frac{1}{5}Ma^2(1+\cos\alpha)(2-\cos\alpha).$$

EXAMPLE 14 *Find the moment of inertia about the x-axis of the portion of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, *which lies in the positive octant, supposing the law of volume density to be* $\rho = \mu xyz$.

Solution Consider an elementary volume $\delta x\, \delta y \delta z$ at a point (x, y, z).

Distance of this element from x-axis $= \sqrt{(y^2+z^2)}$

$\therefore$ moment of inertia of element about x-axis

$$= \rho(y^2+z^2)\,\delta x\,\delta y\,\delta z = \mu xyz\,(y^2+z^2)\,\delta x\,\delta y\,\delta z$$

Hence, the moment of inertia of the octant of the ellipsoid

$$= \iiint \mu\, xyz\,(y^2+z^2)\,dx\,dy\,dz \quad \text{where } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

$\therefore$ required M.I. $= \mu \iiint \frac{1}{8}a^2b^2c^2 (b^2 v + c^3 w)\,du\,dv\,dw$

where $u+v+w \le 1$

Put $\frac{x^2}{a^2} = u, x^2 = a^2 u, x\,dx = \frac{1}{2}a^2 du$, etc.

$$= \frac{1}{8}\mu a^2b^2c^2 \iiint (b^2 u^{1-1} v^{2-1} w^{1-1} + c^2 u^{1-1} v^{1-1} w^{2-1})\,du\,dv\,dw$$

$$= \frac{1}{8}\mu a^2b^2c^2\left[b^2 \frac{\Gamma 1\,\Gamma 2\,\Gamma 1}{\Gamma(1+2+1+1)} + c^2 \frac{\Gamma 1\,\Gamma 1\,\Gamma 2}{\Gamma(1+1+2+1)}\right]$$

$$= \frac{1}{8}\mu a^2b^2c^2 (b^2+c^2)\frac{\Gamma 2}{\Gamma 5} = \frac{1}{8}\mu a^2b^2c^2\,\frac{(b^2+c^2)}{24}$$

Also M = mass of the octant of the ellipsoid.

$$= \iiint \mu\, xyz\,dx\,dy\,dz \quad \text{where } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1$$

$$= \frac{1}{8}\mu a^2b^2c^2 \iiint du\,dv\,dw \quad \text{where } u+v+w \le 1$$

$$= \frac{1}{8}\mu a^2b^2c^2 \frac{\Gamma 1\Gamma 1\Gamma 1}{\Gamma 4} = \frac{1}{8}\mu a^2b^2c^2\cdot\frac{1}{6}$$

Hence, M.I. $= \frac{1}{4}M(b^2+c^2)$.

1.4 THEOREM OF PARALLEL AXIS

Given the moments and products of inertia about axes through the centre of gravity, to find the moments and products of inertia about parallel axes.

Let G be the centre of gravity and GX, GY, GZ, the three axes through G. Take OX', OY', OZ' three parallel axes through any point O.

Let co-ordinates of an element of mass m be (x, y, z) and (x', y', z') referred to two sets of axes.

If $(\bar{x}, \bar{y}, \bar{z})$ are the co-ordinates of G referred to OX', OY', and OZ' as axes, we have

$$x' = x + \bar{x},\ y' = y + \bar{y},\ z' = z + \bar{z}$$

$\therefore$ moment of inertia of the body about OX

$$= \Sigma m(y'^2 + z'^2) = \Sigma m[(y+\bar{y})^2 + (z+\bar{z})^2]$$

$$= \Sigma m(y^2 + z^2 + 2y\bar{y} + 2z\bar{z} + \bar{y}^2 + \bar{z}^2)$$

$$= \Sigma m(y^2+z^2) + 2\bar{y}\,\Sigma my + 2\bar{z}\,\Sigma mz + \Sigma m(\bar{y}^2 + \bar{z}^2);$$

Now $\frac{\Sigma my}{\Sigma m} = 0$, being y-co-ordinate of C.G. referred to G as origin.

$\therefore$ $\Sigma my = 0,$

Similarly, $\Sigma mz = 0.$

Hence, moment of inertia of the body about OX'

$$= \Sigma m(y^2+z^2) + \Sigma m(\dot{y} + \dot{y})$$

$= $ Moment of inertia about $GX + (\bar{y}^3 + \bar{z}^2)\,\Sigma m$

$= $ Moment of inertia about $GX + (\bar{y} + \bar{z}^2)\,M$ as $\Sigma m = M$

$= $ Moment of inertia about GX

+ moment of inertia of a mass M placed at G about OX'.

Also, product of inertia about axes OX', OY'

$$= \Sigma mx'y' = \Sigma m(x+\bar{x})(\bar{y}+\bar{z})$$

$$= \Sigma myx + \bar{x}\,\Sigma my + \bar{y}\,\Sigma mx + \bar{x}\,\bar{y}\,\Sigma m$$

$= \Sigma mxyz + M\bar{x}\,\dot{y}$, the middle two terms vanish

$= $ the product of inertia about GX, GY + the product of inertia of a mass M placed at G about the axes OX', OY'.

SOLVED EXAMPLES

EXAMPLE 1 *Find the moment of inertia of a right circular cylinder about*

(i) *its axis,*

(ii) *a straight line through its centre of gravity perpendicular to its axis.*

Solution (i) Let h be the height and a the radius of base of the cylinder, then mass M of the cylinder is given by

$$m = \pi a^2 h\rho.$$

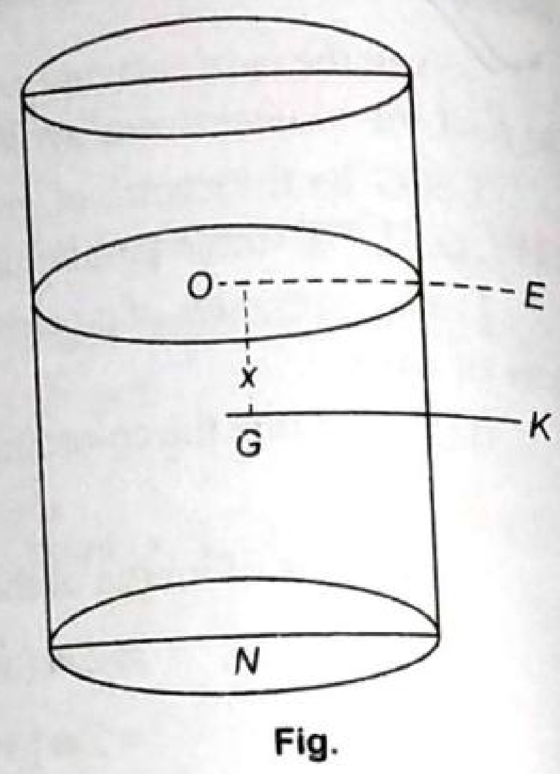

Fig.

To determine moment of inertia about axis ON.

Consider any elementary disc of breadth δx at a distance x from the centre of gravity G.

Moment of inertia of this disc about ON (a line $\perp$ to the disc through its C.G.)

$$= (\pi a^2 \rho \delta x)\frac{a^2}{2}.$$

$\therefore$ M.I. of the whole cylinder about its axis

$$= \int_{-h/2}^{h/2} \pi a^2 \rho dx \cdot \frac{a^2}{2} = \frac{\rho \pi a^4}{2}[x]_{-h/2}^{h/2}$$

$$= \frac{\rho \pi \alpha^4}{2} h = \frac{Ma^2}{2}$$

(ii) To determine moment of inertia about a line through centre of gravity and perpendicular to axis, *i.e.*, about GK.

M.I. of the elementary disc about GK

= M.I. of the disc about OE + M.I. of the mass of the disc, placed at O, about GK

$$= (\pi a^2 \rho \delta x)\frac{a^4}{4} + (\pi a^2 \rho \delta x)\, x^2 = \pi a^2 \rho \left[\frac{a^2}{4} + x^2\right]\delta x$$

$\therefore$ M.I. of the whole cylinder about GK

$$= \pi a^2 \rho \int_{-h/2}^{h/2}\left(\frac{a^2}{4} + x^2\right)dx$$

$$= \pi a^2 \rho \left[\frac{a^2}{4}h + \frac{h^3}{3.4}\right] = \frac{\pi a^2 \rho h}{4}\left(a^2 + \frac{h^2}{3}\right) = \frac{M}{4}\left(a^2 + \frac{h^2}{3}\right).$$

EXAMPLE 2 *Find the moment of inertia of a rectangular parallelopiped about an edge.*

Solution Let $2a, 2b, 2c$ be the lengths of edges of the rectangular parallelopiped.

To determine the moment of inertia about an edge say OA.

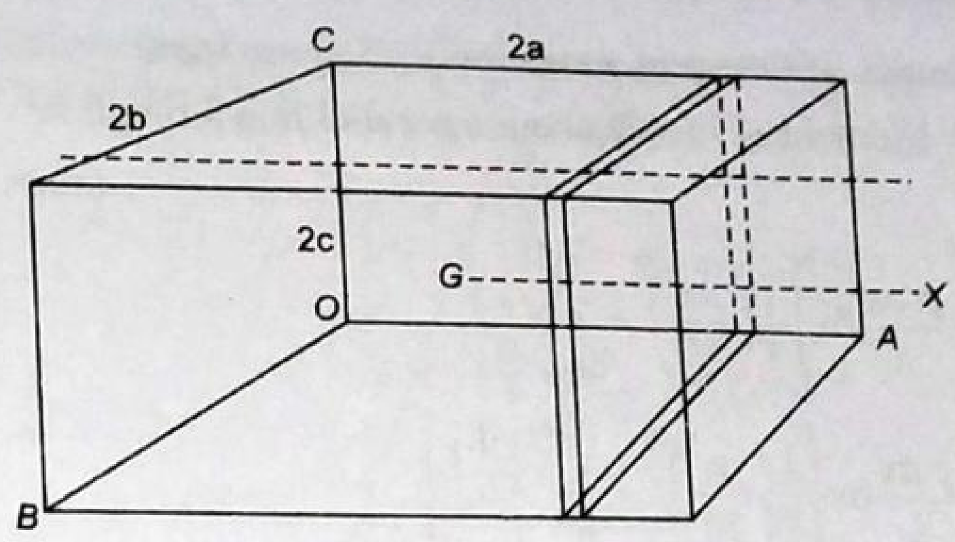

Fig.

M.I. of the rectangular parallelopiped about an axis through C.G. and parallel to OA.

$$= M \cdot \frac{b^2 + c^2}{3}$$

Now M.I. about OA = M.I. of the rectangular parallelopiped about GX

+ M.I. of the mass of the rectangular parallelopiped placed at G, about OA

$$= M\frac{b^2 + c^2}{3} + M(b^2 + c^2) = \frac{4}{3}(b^2 + c^2).$$

Aliter. Referred to O as origin take an element $\delta x\, \delta y\, \delta z$ at a point (x, y, z), then

$$\text{M.I. about } OA = \int_0^{2a}\int_0^{2b}\int_0^{2c} (y^2 + z^2)\rho\, dx\, dy\, dz$$

$$= \rho\left[\frac{y^3}{3}xz + \frac{z^3}{3}xy\right]_{x=0,\ y=0,\ z=0}^{x=2a,\ y=2b,\ z=2c}$$

$$= \frac{32\rho\, abc}{3}[b^2 + c^2]$$

$$= \frac{4}{3}M(b^2 + c^2) \qquad (\text{as } M = 8\, abc\,\rho)$$

EXAMPLE 3 *Find the moment of inertia of the triangle ABC about a perpendicular to the plane through A.*

Solution Let h be the height of the triangle.

Consider an elementary strip PQ of breadth δx at a distance $x\,(= AN)$ from A. AK is an axis perpendicular to the lamina.

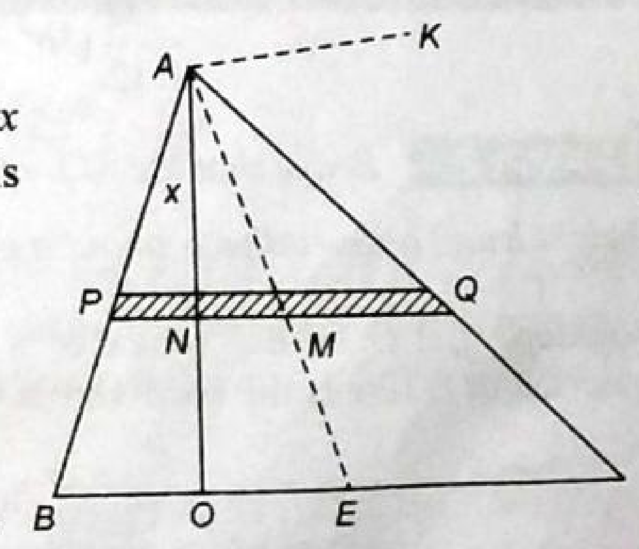

Fig.

Also $\dfrac{x}{h} = \dfrac{PQ}{BC}$;

$\therefore$ $PQ = \dfrac{ax}{h}$.

Moment of inertia of this strip about AK

= Moment of inertia about a parallel line through M + (mass of strip). AM^2

$$= \frac{\rho\, ax}{h}\delta x\left[\frac{1}{3}\left(\frac{ax}{2h}\right)^2 + AM^2\right]$$

$$= \frac{\rho\, ax}{h}\delta x\left[\frac{1}{3}\left(\frac{ax}{2h}\right)^2 + \frac{x^2}{h^2}AE^2\right] \qquad \left(\text{as } \frac{AM}{AE} = \frac{x}{h}\right)$$

$\therefore$ required moment of inertia $= \dfrac{\rho a}{4h^3}\displaystyle\int_0^h \left[\frac{a^2}{3} + 4AE^2\right]x^3\,dx$

$$= \frac{\rho a}{4h^3}\left(\frac{a^2}{3} + 4AE^2\right)\frac{h^4}{4}$$

$$= \frac{\rho\, ah}{48}[a^2 + 12AE^2]$$

$$= \frac{M}{24}[a^2 + 12AE^2] \qquad \left(\text{as } M = \frac{1}{2}ah\rho\right)$$

But $\quad AE^2 = AO^2 + OE^2 = AO^2 + (BE - BO)^2$

$$= (AO^2 + BO^2) + BE^2 - 2BE \cdot BO$$

$$= AB^2 + \left(\frac{1}{2}BC\right)^2 - 2\left(\frac{1}{2}BC\right)AB\cos B$$

$$= c^2 + \frac{a^2}{4} - ac\cdot\frac{a^2 + c^2 - b^2}{2ac} = \frac{2b^2 + 2c^2 - a^2}{4}$$

Hence, moment of inertia becomes

$$= \frac{M}{24}[a^2 + 3(2b^2 + 2c^2 - a^2)]$$

$$= \frac{M}{12}[3b^2 + 3c^2 - a^2]$$

EXAMPLE 4 *Prove that the M.I. of a uniform right circular solid cone of mass M, height h and base-radius r, about a diameter of its base is* $\dfrac{M}{20}(3r^2 + 2h^2)$.

Solution Let O be the vertex of a right circular cone of mass M, height h and base-radius r. If α is the semi-vertical angle of ρ the density of the cone.

Then, $\quad M = \dfrac{1}{3}\pi h^3 \rho \tan^2\alpha \qquad \ldots(1)$

Consider an elementary disc EF of thickness δx, parallel to the base AB and at a distance x from the vertex O.

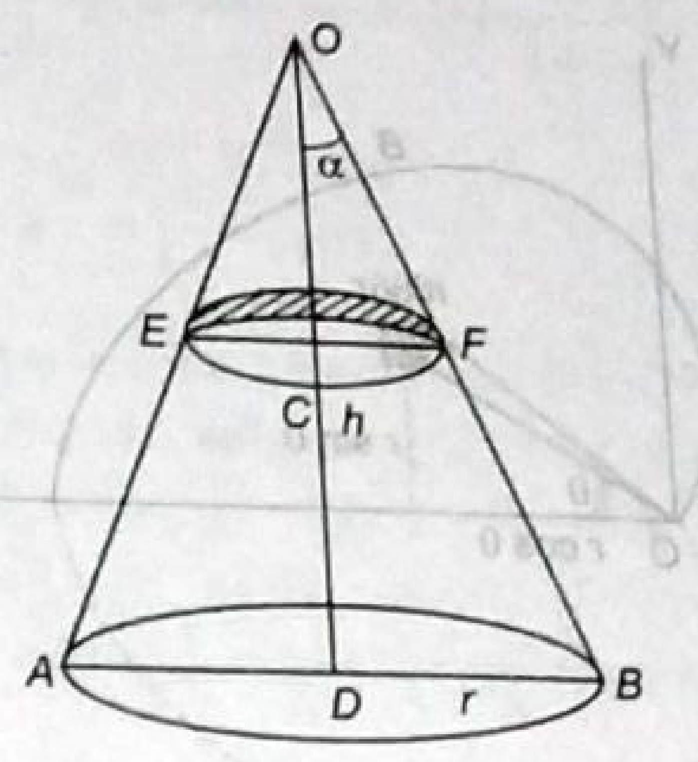

Fig.

$\therefore$ Mass of the disc $= \delta m = \rho\pi x^2 \tan^2\alpha\,\delta x$.

$\therefore$ M.I. of the disc about the diameter AB of the base of the cone

= Its M.I. about parallel diameter EF of the disc

+ M.I. of the total mass δm at centre C about AB

$$= \frac{1}{4}\delta m\cdot CE^2 + \delta m\cdot CD^2 = \rho\pi x^2\tan^2\alpha\left[\frac{1}{4}x^2\tan^2\alpha + (h-x)^2\right]\delta x$$

$$= \int_0^h \rho\pi x^2\tan^2\alpha\left[\frac{1}{4}x^2\tan^2\alpha + (h-x)^2\right]dx$$

$$= \frac{1}{4}\rho\pi\tan^2\alpha\int_0^h (x^4\tan^2\alpha + 4h^2x^2 - 8hx^3 + 4x^4)\,dx$$

$$= \frac{1}{4}\rho\pi\tan^2\alpha\left[\frac{1}{5}h^5\tan^2\alpha + \frac{4}{3}h^5 - 2h^5 + \frac{4}{5}h^5\right]$$

$$= \frac{1}{60}\rho\pi h^5\tan^2\alpha\,[3\tan^2\alpha + 2] \qquad \text{[using Eq. (1)]}$$

$$= \frac{1}{20}Mh^2\,[3\tan^2\alpha + 2]$$

$$= \frac{1}{20}Mh^2\left(3\cdot\frac{a^2}{h^2} + 2\right) = \frac{M}{20}(3a^2 + 2h^2).$$

EXAMPLE 5 *A solid body, of density ρ, is in the shape of the solid formed by the revolution of the cardioid* $r = a(1 + \cos\theta)$ *about the initial line; show that its moment of inertia about a straight line through the pole perpendicular to the initial line is* $\dfrac{352}{105}\pi\rho a^2$.

Solution OY is the line through the pole perpendicular to the initial line.

To find the moment of inertia of the body (formed by the revolution of area OBA about OA) about OY.

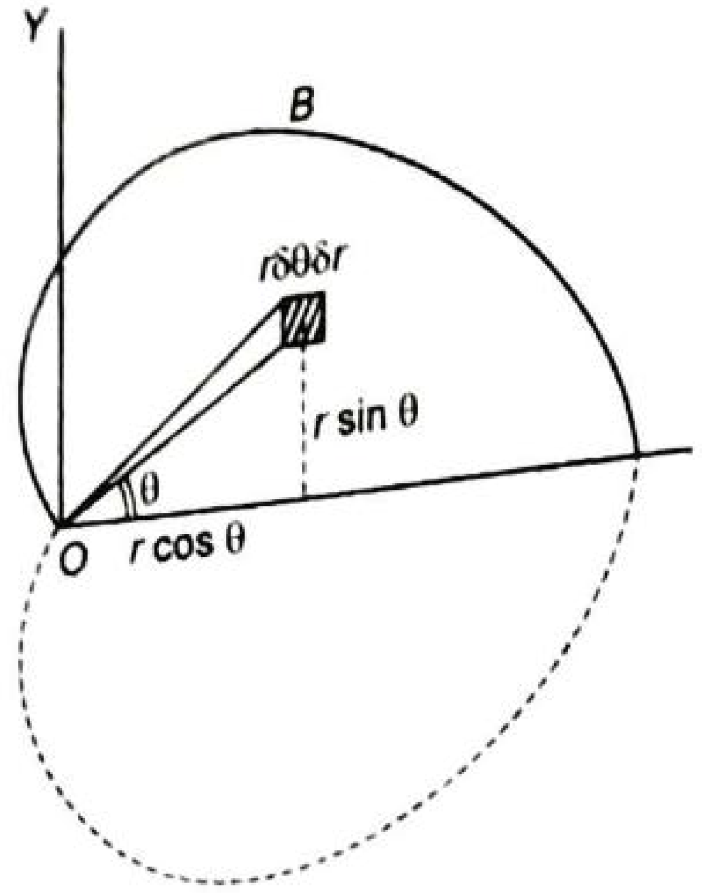

Fig.

Consider an elementary area $r\,\delta\theta\,\delta r$; this when revolved about OA forms a circular ring of radius $r\sin\theta$.

M.I. of the ring about OY (a diameter)

$$= (2\pi r \sin\theta\, r\,\delta\theta\,\delta r\,\rho)\frac{r^2\sin^2\theta}{2}$$

$\therefore$ M.I. of the ring about $OY = (2\pi r\sin\theta\, r\,\delta\theta\,\delta r\rho)\left(\dfrac{r^2\sin^2\theta}{2} + r^2\cos^2\theta\right)$

Hence, moment of inertia of the whole solid of revolution about OY

$$= 2\pi\rho\int_0^\pi\int_0^{a(1+\cos\theta)} r^4\sin\theta\left(\frac{\sin^2\theta}{2}+\cos^2\theta\right)d\theta\,dr$$

$$= \pi\rho\int_0^\pi\int_0^{a(1+\cos\theta)} r^4\sin\theta\,(1+\cos^2\theta)\,d\theta\,dr$$

$$= \frac{\pi\rho a^5}{5}\int_0^\pi (1+\cos\theta)^5\sin\theta\,(1+\cos^2\theta)\,d\theta$$

$$= \frac{\pi\rho a^5}{5}\int_0^\pi\left(2\cos^2\frac{\theta}{2}\right)^5\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\times\left[1+\left(2\cos^2\frac{\theta}{2}-1\right)^2\right]d\theta$$

$$= \frac{\pi\rho a^5\cdot 2^6}{5}\int_0^\pi\left[2\cos^{11}\frac{\theta}{2}\sin\frac{\theta}{2}+4\sin\frac{\theta}{2}\cos^{15}\frac{\theta}{2}-4\sin\frac{\theta}{2}\cos^{13}\frac{\theta}{4}\right]d\theta$$

$$= \frac{\pi\rho a^5 2^8}{5}\int_0^{\frac{1}{2}\pi}[\cos^{11}t\sin t+2\cos^{15}t\sin t-2\cos^{13}t\sin t]\,dt,$$

where $t = \dfrac{\theta}{2}$.

$$= \frac{\pi\rho a^5 2^8}{5}\left[-\frac{\cos^{12}t}{12}-2\cdot\frac{\cos^{16}t}{16}+2\cdot\frac{\cos^{14}t}{14}\right]_0^{\frac{1}{2}\pi}$$

$$= \frac{\pi\rho a^5 2^8}{5}\left[\frac{1}{12}+\frac{1}{8}-\frac{1}{7}\right] = \frac{352\pi\rho a^5}{105}.$$

EXAMPLE 6 *A closed curve revolves round any line OX in its own plane which does not intersect it, show that the moment of inertia of the solid of revolution so formed about OX is equal to $M(a^2+3k^2)$, where M is the mass of the solid generated, a is the distance from OX of the centre C of the curve, and k is the radius of gyration of the curve about a line through C parallel to OX.*

Solution Let C be centre of the central curve and S be its area.

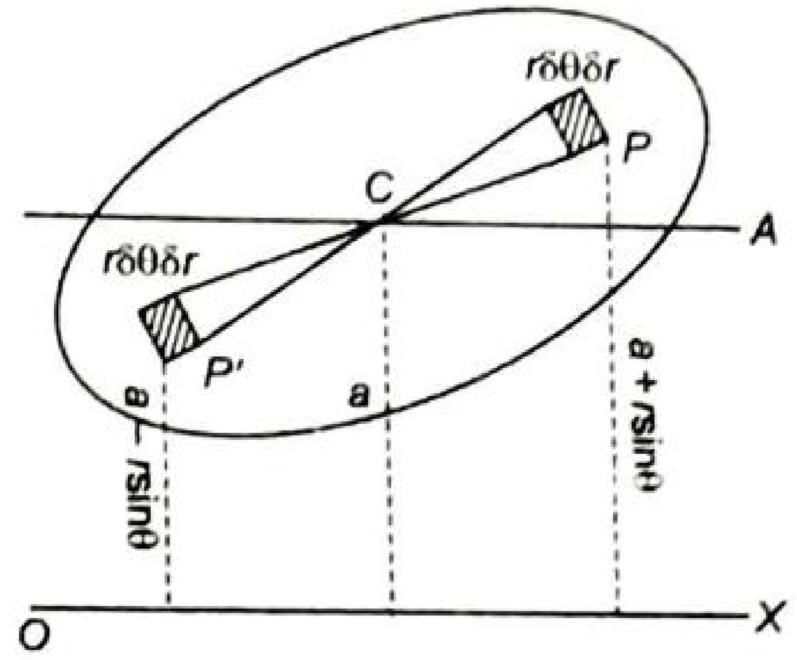

Fig.

Take a line CA parallel to OX at a distance a from OX. If ρ is the density and M the mass of the solid formed, then

$$M = 2\pi a\rho S \qquad \ldots(1)$$

Consider an element $r\,d\theta\,\delta r$ at a distance r from C, making an angle θ with CA. Since the curve is a central curve, it will have an equal element for the same value θ in the opposite direction.

Distances of these elements from OX are respectively

$$a + r\sin\theta \quad\text{and}\quad a - r\sin\theta$$

Now $$S = \iint 2r\,d\theta\,dr, \qquad \ldots(2)$$

integration being taken to cover the upper half of the area.

Moment of inertia of the area S about CA is $S\rho k^2$.

$\therefore$ $$S\rho k^2 = \iint \rho\cdot 2r\,d\theta\,r^2\sin\theta \qquad \ldots(3)$$

Now moment of inertia of the solid of revolution about OX

$$= \iint r\,d\theta\cdot dr\,\rho\,\{2\pi(a+r\sin\theta)(a+r\sin\theta)^2$$
$$+ 2\pi(a-r\sin\theta)(a-r\sin\theta)^2\}$$

$$= 2\pi\rho \iint \{(a + r\sin\theta)^3 + (a - r\sin\theta)^3\}\, r\, d\theta\, dr$$

$$= 4\pi\rho \iint r(a^3 + 3ar^2\sin^2\theta)\, d\theta\, dr$$

$$= 2\pi\rho a^3 \iint 2r\, d\theta\, dr + 2\pi\rho a \cdot 3 \iint 2r\, d\theta\, dr\, r^2\sin^2\theta \quad \text{[from Eqs. (2) and (3)]}$$

$$= 2\pi a^3\rho S + 2\pi a\rho \cdot 3Sk^2 \quad \text{[from Eq. (1)]}$$

$$= Ma^2 + 3Mk^2 = M(a^2 + 3k^2)$$

EXAMPLE 7 *Prove a theorem similar to the one proved in Ex. 6 for the moment of inertia of the surface generated by the arc of the curve.*

Solution Let l be the length of the whole curve

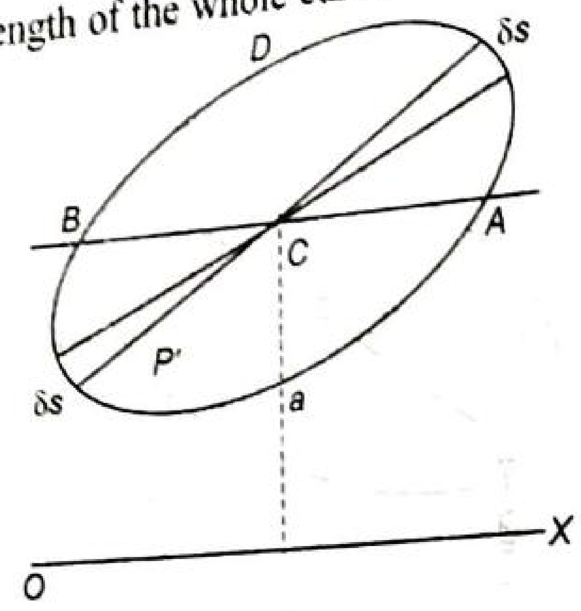

Fig.

Then, $$l = 2\int ds, \quad \ldots(1)$$

integration being taken to cover the upper half of the curve.

$$M = \text{mass of the surface of revolution}$$

$$= 2\pi a\rho l, \quad \ldots(2)$$

If k is the radius of gyration of the arc of the curve about CA,

$$\rho l k^2 = \text{M.I. of the arc about } CA$$

$$= 2\int \rho r^2 \sin^2\theta\, ds \quad \ldots(3)$$

Now arguing as in the above example, M.I. of the surface of revolution about OA

$$= \int \rho[2\pi(a + r\sin\theta)^3 + 2\pi(a - r\sin\theta)^3]\, ds$$

$$= 4\pi\rho \int (a^3 + 3ar^2\sin^2\theta)\, ds$$

$$= 2\pi\rho a^3 \int 2ds + 6\pi\rho a \int 2r^2\sin^2\theta\, ds$$

$$= Ma^2 + 2Mk^2 \quad \text{from Eqs. (1), (2) and (3)}$$

$$= M(a^2 + 3k^2)$$

EXAMPLE 8 *The moment of inertia about its axis, of a solid rubber tyre, of mass M and circular cross section of radius a is*

$$\frac{M}{4}(4b^2 + 3a^2), \qquad \textit{where } b \textit{ is the radius of the curve.}$$

If the tyre be hollow and of small uniform thickness, show that the corresponding result is

$$\frac{M}{2}(2b^2 + 3a^2).$$

Solution This is a particular case of the above two examples. The tyre is formed by the revolution of a circular area about an axis.

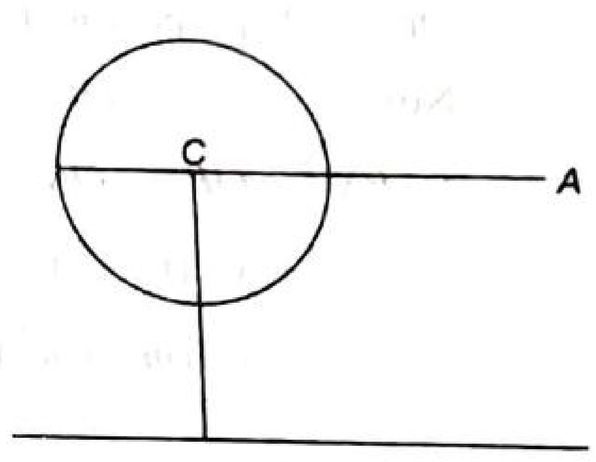

Fig.

Case I. For solid tyre. (It is case of Ex. 5).

Moment of inertia of circular area about CA

$$= \text{mass} \times \frac{a^2}{4}$$

$$\therefore \quad k^2 = \frac{a^2}{4},$$

and put b for a in the result of ex. 5.

Hence, $$\text{required M.I.} = M\left(b^2 + \frac{3a^2}{4}\right) = \frac{M}{4}(4b^2 + 3a^2).$$

Case II. Hollow tyre. [It is case of Ex. 6].

Moment of inertia of circular arc about $GA = \text{mass} \cdot \frac{a^2}{2}$

So, here $$k^2 = \frac{a^2}{2}.$$

Thus, putting b for a in result of Ex. 7.

$$\text{M.I. of the hollow tyre} = M\left(b^2 + \frac{2a^3}{2}\right) = \frac{M}{2}(2b^2 + 3a^2).$$

1.5 MOMENT OF INERTIA ABOUT A LINE

Given the moments and products of inertia about three perpendicular axes, to find the moment of inertia about any line through their meeting point.

Let A, B, C be the moment of inertia about the three given axes OX, OY, OZ. Also let D, E, F be the products of inertia with respect to the axes of y and z, z and x, x and y respectively.

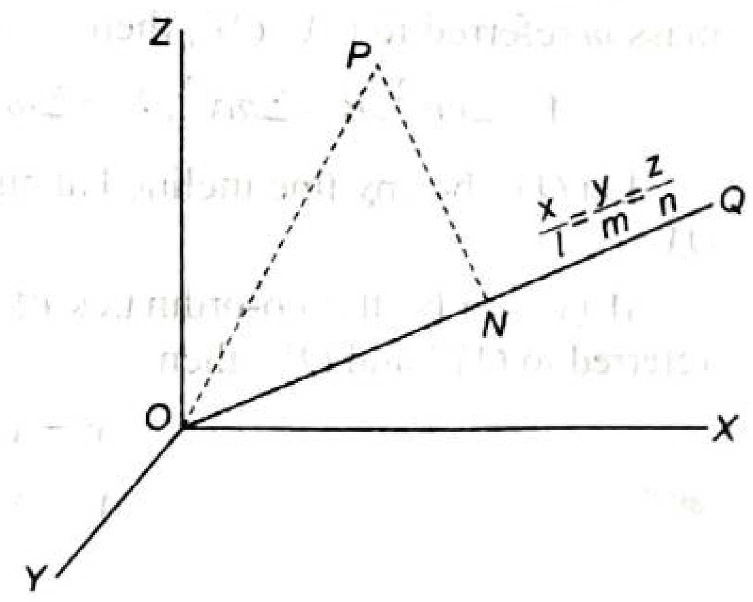

Fig.

Now if m be the element of mass at P whose co-ordinates are (x, y, z), then

$$A = \Sigma m' (y^2 + z^2), \quad C = \Sigma m' (x^2 + y^2) \qquad \ldots(1)$$
$$B = \Sigma m' (x^2 + z^2),$$
$$D = \Sigma m' yz, \quad E = \Sigma m' zx \text{ and } F = \Sigma m' xy$$

Let OQ be a line whose direction-cosines are (l, m, n). We are required to find moment of inertia of the body about OQ.

Draw PN perpendicular from P on OQ.

Now $\quad OP^2 = x^2 + y^2 + z^2, \quad ON = lx + my + nz.$

$$\therefore \; PN^2 = OP^2 - ON^2 = (x^2 + y^2 + z^2) - (lx + my + nz)^2$$
$$= x^2 (1 - l^2) + y^2 (1 - m^2) + z^2 (1 - n^2) - 2mnyz - 2lnzx + 2lmxy$$
$$= x^2 (m^2 + n^2) + y^2 (l^2 + n^2) + z^2 (l^2 + m^2) - 2mnyz - 2lnzx - 2lmxy$$
$$\text{as} \quad l^2 + m^2 + n^2 = 1$$
$$= l^2 (y^2 + z^2) + m^2 (x^2 + z^2) + n^2 (x^2 + y^2) - 2mnyz - 2lnzx - 2lmxy$$

If I denotes the moment of inertia of the body about OQ, then

$$I = \Sigma m' \cdot PN^2 = l^2 \Sigma m' (y^2 + z^2) + m^2 \Sigma m' (x^2 + z^2) + n^2 \Sigma m' (x^2 + y^2)$$
$$- 2mn \Sigma m' yz - 2ln \Sigma m' zx - 2lm \Sigma m' xy$$
$$= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Eln - 3Flm \qquad \text{[from Eq. (1)]}$$

1.6 AN IMPORTANT RESULT

If the moments and product of inertia of a plane lamina about two perpendicular axes in the plane are known; to find the moment of inertia about any other axis through their point of intersection.

Let A and B are the moments and F the product of inertia of a plane lamina about two axes OX and OY in the plane.

If (x, y) be the co-ordinates of the element of mass m referred to OX, OY, then

$$A = \Sigma my^2, B = \Sigma mx^2, F = \Sigma mxy \qquad \ldots(1)$$

Let OX' be any line inclined at an angle α to OX.

If (x', y') be the co-ordinates of element m referred to OX' and OY' then

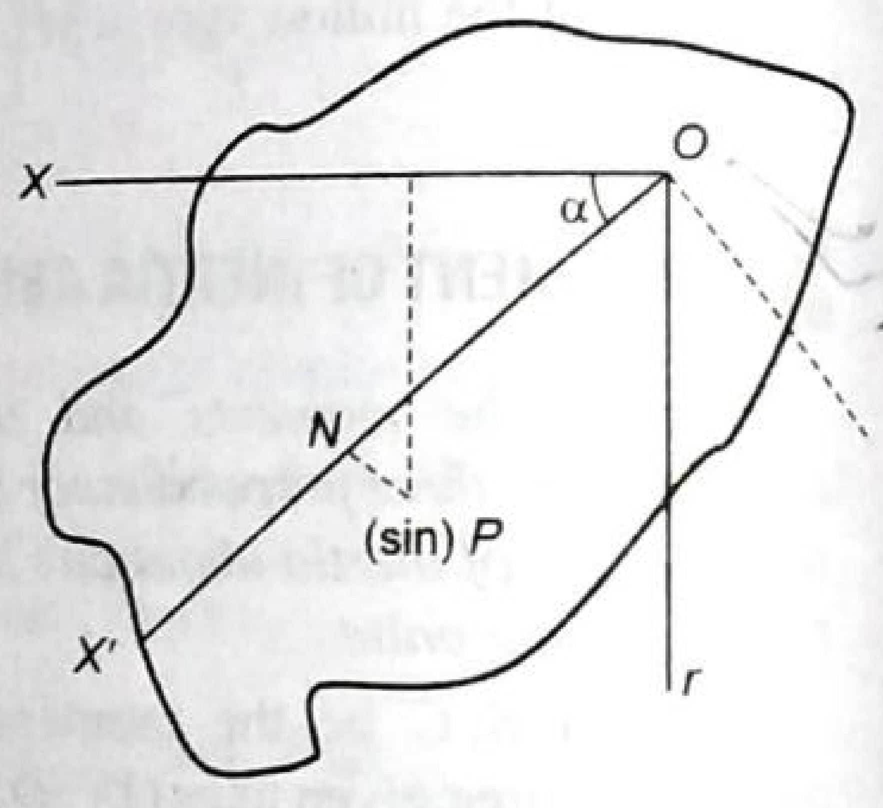

Fig.

$$x' = x \cos\alpha + y \sin\alpha$$

and
$$y' = y \cos\alpha - x \sin\alpha$$

Now moment of inertia of the body about OX'

$$= \Sigma my'^2$$
$$= \Sigma m (y \cos\alpha - \sin\alpha)^2$$
$$= \cos^2\alpha \; \Sigma my^2 - 2 \sin\alpha \cos a \; \Sigma mxy + \sin^2\alpha \; \Sigma mx^2$$
$$= A \cos^2\alpha - 2F \sin\alpha \cos\alpha + A \sin^2\alpha \qquad \text{from Eq. (1)}$$
$$= A \cos^2\alpha + B \sin^2\alpha - F \sin 2\alpha$$

Also, product of inertia about OX', OY'

$$= \Sigma mx' y$$
$$= \Sigma m (x \cos\alpha + y \sin\alpha)(y \cos\alpha - x \sin\alpha)$$
$$= \sin\alpha \cos\alpha \; (\Sigma my^2 - \Sigma mx^2) + (\cos^2\alpha - \sin^2\alpha) \Sigma mxy$$
$$= (A - B) \sin a \cos\alpha + F \cos 2\alpha$$
$$= \frac{1}{2}(A - B) \sin 2\alpha + F \cos 2\alpha.$$

Some Simple Propositions

Proposition I. *If A, B, C stand for moments and D, E, F for the products of inertia about the axes, then the sum of any two of them is greater than the third.*

Proof. We see that $A + B \;\; C = \Sigma m (y^2 + z^2) + \Sigma m (x^2 + z^2) - \Sigma m (x^2 + y^2)$

$$= 2\Sigma mz^2 = +\text{ive}$$

This proves the proposition.

Proposition II. *The sum of the moments of inertia about any three axes (rectangular) meeting at a given point is always constant and is equal to twice the moment of inertia about that point.*

Proof. We see that $\quad A + B + C = 2\Sigma m (x^2 + y^2 + z) = 2\Sigma mr^2,$

which shows that $A + B + C$ is independent of the direction of axes.

Proposition III. *The sum of the moments of inertia of a body with reference to any plane through a given point and its normal at that point is coustant and is equal to the moment of inertia if the body with refrence to that point.*

Proof. Let the given point be taken as origin and plane as the plane of xy.

If C' is the moment of inertia w.r.t. xy plane and C the moment of inertia about its normal at origin (z-axis), then

$$C' + C = \Sigma mr^2 = \frac{1}{2}(A + B + C) \qquad \text{from Proposition II.}$$

which is independent of the direction of the axes.

$$\therefore \qquad C' = \frac{1}{2}(A + B - C).$$

Thus, if A', B', C' are the moments of inertia with reference to the planes of $y, z; z, x$ and x, y, then

$A' = \frac{1}{2}(B + C - A)$, $B' = \frac{1}{2}(C + A - B)$, $C' = \frac{1}{2}(A + B - C)$.

Proposition IV. *We have to show that*

$A > 2D$, $B > 2E$ and $C > 2F$

Proof. We see that $(y^2 + z^2) > 2yz$, etc. $(\because$ A.M. > G.M.$)$

SOLVED EXAMPLES

EXAMPLE 1 *Show that moment of inertia of a rectangle of mass M and sides 2a, 2b about a diagonal is* $\frac{2M}{3}\frac{a^2b^2}{a^2+b^2}$.

Solution $ABCD$ is the rectangle, with sides

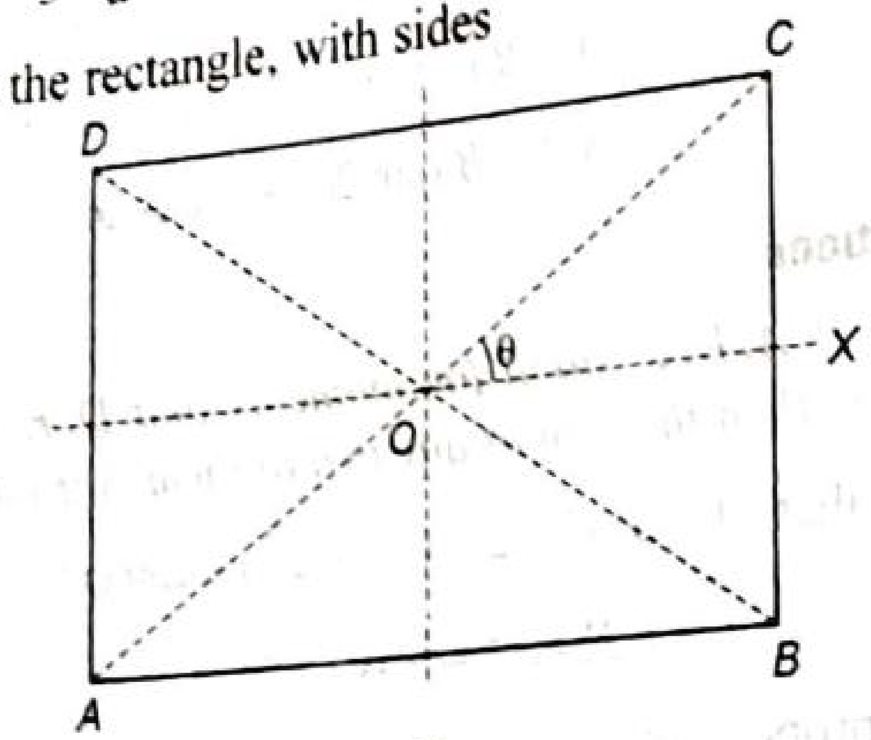

Fig.

$$AB = 2a,\ AD = 2b.$$

Its moment of inertia about $OX = \frac{1}{3}Mb^2$.

Its moment of inertia about $OY = \frac{1}{3}Ma^2$.

Let the diagonal AC make an angle θ with AB,

then $\tan\theta = \frac{b}{a}$ so that $\sin\theta = \frac{b}{\sqrt{(a^2+b^2)}}$ and $\cos\theta = \frac{a}{\sqrt{(a^2+b^2)}}$.

Thus, moment of inertia about AC

$$= \frac{1}{3}Mb^2\cos^2\theta + \frac{1}{3}Ma^2\sin^2\theta \qquad \ldots(1)$$

$$= \frac{Mb^2}{3}\cdot\frac{a^2}{a^2+b^2} + \frac{Ma^2}{3}\cdot\frac{b^2}{a^2+b^2} = \frac{2Ma^2b^2}{3(a^2+b^2)}$$

Note. Here Eq. (1) in general gives the moment of inertia of the rectangle about a line through its centre and making an angle θ with side 2a.

EXAMPLE 2 *Show that the moment of inertia of right solid cone whose height is h and radius of whose base is a, is* $\frac{3Ma^2}{20}\cdot\left(\frac{6h^2+a^2}{h^2+a^2}\right)$ *about a slant side, and* $\left(\frac{3M}{80}\right)(h^2+4a^2)$ *about a line through the centre of gravity of the cone perpendicular to its axis.*

Solution Let α be the semi-vertical angle of the cone; then

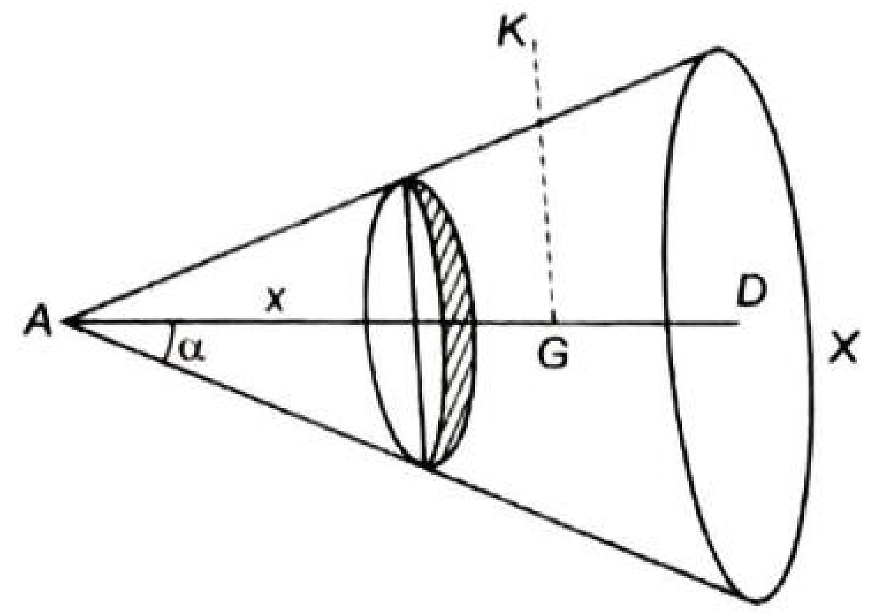

Fig.

$$\tan\alpha = \frac{a}{h}. \qquad \ldots(1)$$

M is the mass of the cone.

$$\therefore \quad M = \frac{1}{3}\pi a^2 h\rho \qquad \ldots(2)$$

Consider an elementary disc of thickness δx at a distance x from A;

radius of the disc $= x\tan\alpha$

mass of the disc $= \rho\pi x^2\tan^2\alpha\,\delta x$.

$\therefore$ M.I. of the cone about $AD = \rho\pi x^2\tan^2\alpha\cdot\frac{x^2\tan^2\alpha}{2}\delta x$.

$$= \frac{1}{2}\rho\pi\tan^4\alpha\, x^4\delta x.$$

Hence, M.I. of the cone about $AD = \frac{\rho\pi\tan^4\alpha}{2}\int_0^h x^4\,dx$

$$= \frac{\pi\tan^4\alpha\,\rho h^5}{10} = \frac{3Ma^2}{10} \qquad \ldots(3)$$

Also M.I. of the cone about AE (a line through vertex A perpendicular to AD)

$$= \int_0^h \pi x^2\tan^2\alpha\cdot\rho\,dx\left[\frac{x^2\tan^2\alpha}{4} + x^2\right]$$

$$= \pi\tan^2\alpha.\rho\int_0^h\left(\frac{\tan^2\alpha}{4}+1\right)x^4dx$$

$$= \frac{\pi\tan^2\alpha\,\rho h^5}{4}\left(\frac{\tan^2\alpha}{4}+1\right)$$

$$= \frac{\pi}{20}\cdot\frac{a^2}{h^2}\frac{3Mh^5}{\pi a^2 h}\left(\frac{a^2}{h^2}+4\right) \quad \text{as } M = \frac{1}{3}\pi a^2 h\rho \text{ from Eq. (2)}$$

$$= \frac{3M}{20}(a^2+4h^2) \qquad \ldots(4)$$

Product of inertia of the cone about AX, AE is clearly zero.

Now moment of inertia of the cone about slant height

$$= \frac{3Ma^2}{10}\cos^2\alpha + \frac{3M}{20}(a^2+4h^2)\sin^2\alpha$$

$$= \frac{3Ma^2}{10}\cdot\frac{h^2}{a^2+h^2} + \frac{3M}{20}(a^2+4h^2)$$

$$= \frac{3Ma^2}{20}\cdot\frac{6h^2+a^2}{a^2+h^2}$$

as $\tan\alpha = \frac{a}{h}, \sin\alpha = \frac{a}{\sqrt{(a^2+h^2)}}$

and $\cos\alpha = \frac{h}{\sqrt{(a^2+h^2)}}$

Part II. *To find the moment of inertia about a line (GK) through centre of gravity perpendicular to the axis.*

M.I. about AE = M.I. about GK + M.I. about AE of Mass M placed at G

∴ M.I. about GK = M.I. about AE − M.I. about AE of mass M paced at G.

$$= \frac{\pi a^2\rho h^3}{5}\left[\frac{a^2}{4h^2}+1\right] - M\cdot\frac{9h^2}{16} = \frac{3M}{5}\left(\frac{a^2}{4}+h^2\right) - M\cdot\frac{9h^2}{16}$$

$$= \frac{3M}{80}[4a^2+16h^2-15h^2]$$

$$= \frac{3M}{80}[h^2+4a^2]$$

EXAMPLE 3 *A closed shell of total mass M, made of thin uniform sheet metal, is in the thin form of a right circular cone, of slant height l and base radius r. Prove that the moment of inertia of the shell about its axis of symmetry is $\frac{1}{2}Mr^2$, and that about a line through the vertex perpendicular to the axis is $\frac{1}{4}M(2l^2+2rl-3r^2)$.*

Solution Let h be the height and α the semi-vertical angle of the cone, such that

$$r = h\tan\alpha,\ l = a\sec\alpha \qquad \ldots(1)$$

If ρ be the density and M the mass of the closed shell, then [See fig. of ex-1]

M = mass of the curved surface + mass of the base

$= \pi(h\tan\alpha)h\sec\alpha\,\rho + \pi h^2\tan^2\alpha\,\rho$

$= \pi h^2\rho\tan\alpha\,(\sec\alpha+\tan\alpha) \qquad \ldots(2)$

Now consider a circular ring at a distance x from the vertex A, so that width of the ring is $\delta x\sec\alpha$.

Now M.I. of the closed conical shell about AX

= M.I. of the hollow cone about AX + M.I. of the base about AX

$$= \int_0^h (2\pi x\tan\alpha)\rho\,dx\sec\alpha\cdot x^2\tan^2\alpha + (\pi h^2\tan^2\alpha\,\rho)\frac{h^2\tan^2\alpha}{2}$$

$$= 2\pi\rho\frac{h^4}{4}\tan^3\alpha\sec\alpha + \frac{1}{2}\rho\pi h^4\tan^4\alpha$$

$$= \frac{1}{2}\pi\rho h^4\tan^3\alpha(\sec\alpha+\tan\alpha) = \frac{1}{2}Mh^2\tan^2\alpha \qquad \text{[from Eq. (2)]}$$

$$= \frac{1}{2}Mr^2 \qquad \text{[from Eq. (1)]}$$

This proves the first result.

Again M.I. about AE (a line perpendicular to AX through A)

= M.I. of the hollow cone about AE + M.I. of the base about AE

$$= \int_0^h (2\pi x\tan\alpha)\rho\,dx\sec\alpha\left(\frac{1}{2}x^2\tan^2\alpha + x^2\right) + \pi h^2\tan^2\alpha\,\rho\left[\frac{h^2\tan^2\alpha}{4}+h^2\right]$$

$$= \frac{h^4}{4}\pi\rho\tan\alpha\sec\alpha\,(\tan^2\alpha+2) + \frac{1}{4}\pi\rho h^4\tan^2\alpha\,(\tan^2\alpha+4)$$

$$= \frac{1}{4}h^4\pi\rho\tan\alpha\,[\tan^2\alpha\,(\sec\alpha+\tan\alpha+2(\sec\alpha+2\tan\alpha)]$$

$$= \frac{1}{8}Mh^2\left[\tan^2\alpha + \frac{2(\sec\alpha+2\tan\alpha)}{\sec\alpha+\tan\alpha}\right] \qquad \text{[from Eq. (2)]}$$

$$= \frac{1}{4}M(h^2\tan^2\alpha + 2h^2(\sec\alpha-\tan\alpha)(\sec\alpha+2\tan\alpha)]$$

$$= \frac{1}{4}M[2h^2\sec^2\alpha + 2h^2\tan\alpha\sec\alpha - 2h^2\tan^2\alpha]$$

$$= \frac{1}{4}M(2l^2+2lr-3r^2]$$

This proves the second result.

EXAMPLE 4 *Show that the moment of inertia of elliptic area of mass M and semi-axis a and b about a diameter of length r is $\frac{1}{4}M\frac{a^2b^2}{r^2}$.*

Solution Moment of inertia of the ellipse about major axis, $CX = \frac{Mb^2}{4}$, and its moment of inertia about minor axis $Cy = \frac{Ma^2}{4}$, also the products of inertia about CX, CY vanishes.

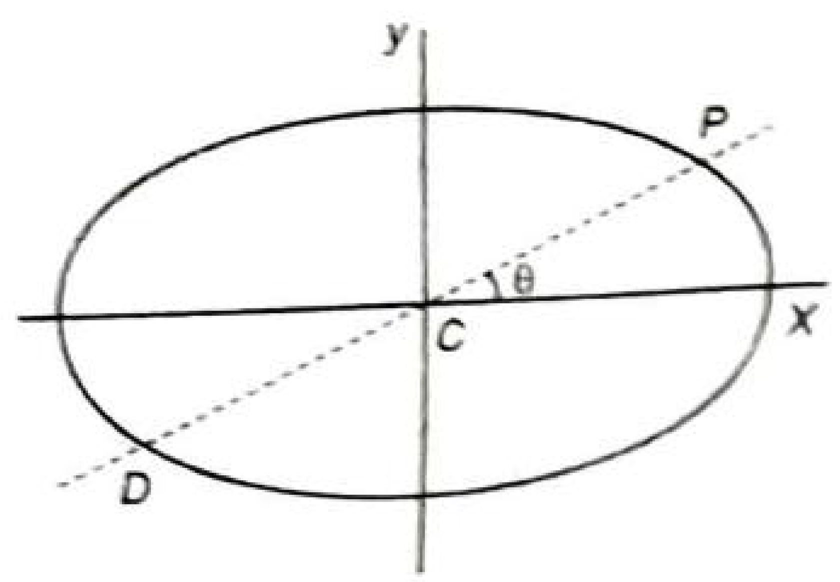

Fig.

Let PD be a diameter making an angle θ with the axis of x, then moment of inertia of the ellipse about diameter CP.

$$= \frac{Mb^2}{4}\cos^2\theta + \frac{Ma^2}{4}\sin^2\theta$$

$$= \frac{M}{4}(b^2\cos^2\theta + a^2\sin^2\theta) \quad \ldots(1)$$

Given r as the length of the semi-diameter CP, so co-ordinates of P are $(r\cos\theta, r\sin\theta)$.

As P is on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$$\therefore \quad \frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{b^2} = 1,$$

i.e., $$\frac{r^2}{a^2b^2}(b^2\cos^2\theta + a^2\sin^2\theta) = 1$$

$$\therefore \quad b^2\cos^2\theta + a^2\sin^2\theta = \frac{a^2b^2}{r^2}$$

Substituting from this in Eq. (1), we have moment of inertia of the elliptic disc about $CP = \frac{M}{4}\cdot\frac{a^2b^2}{r^2}$.

EXAMPLE 5 Show *that the moment of inertia of an ellipse of mass M and semi-axis a and b about a tangent is* $\frac{5M}{4}p^2$, *where p is the perpendicular from the centre on the tangent.*

Solution Let tangent be inclined at an angle θ to be the axis of x, then its equation is

$$y = x\tan\theta + \sqrt{(a^2\tan^2\theta + b^2)},$$

p = perpendicular distance of the tangent from the centre (0, 0)

$$= \frac{\sqrt{a^2\tan^2\theta + b^2)}}{\sqrt{(1+\tan^2\theta)}} = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}$$

Now proceeding as in last example.

Moment of inertia about a diameter parallel to the given tangent

$$= \frac{M}{4}(a^2\sin^2\theta + b^2\cos^2\theta) = \frac{M}{4}p^2 \quad \text{(see Eq. 1 of last Ex.)}$$

Therefore, by theorem on parallel axis required moment of inertia about the tangent

$$= \frac{M}{4}p^2 + Mp^2 = \frac{5}{4}Mp^2.$$

EXAMPLE 6 *If k_1, k_2 be the radii of gyration of an elliptic lamina about two conjugate diameters, then*

$$\frac{1}{k_1^2} + \frac{1}{k_2^2} = 4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Solution Let CP and CD be conjugate semi-diameter of lengths r_1 and r_2; then

$$Mk_1^2 = \frac{M}{4}\cdot\frac{a^2b^2}{r_1^2} \quad \text{and} \quad Mk_2^2 = \frac{M}{4}\cdot\frac{a^2b^2}{r_2^2} \quad \text{(as in Ex. 4)}$$

$$\therefore \quad \frac{1}{k_1^2} + \frac{1}{k_2^2} = \frac{4}{a^2b^2}(r_1^2 + r_2^2)$$

$$= \frac{4}{a^2b^2}(a^2 + b^2) \quad \text{as} \quad r_1^2 + r_2^2 = a^2 + b^2 \text{ by property}$$

$$= 4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

EXAMPLE 7 *Show that the sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same.*

Solution M.I. about a tangent inclined at an angle θ

$$= \frac{5}{4}Mp^2 \quad \textbf{(found in Ex. 5)}$$

$$= \frac{5}{4}M(a^2\sin^2\theta + b^2\cos^2\theta) \quad [\text{as } p = \sqrt{(a^2\sin^2\theta + b^2\cos^2\theta)}]$$

It follows then that moment of inertia about a perpendicular tangent

$$= \frac{5}{4}M(a^2\cos^2\theta + b^2\sin^2\theta) \quad \left\{\text{putting } \frac{\pi}{2} + \theta \text{ for } \theta\right\}$$

$\therefore$ Sum of the moments of inertia about two perpendicular tangents $= \frac{5}{4}M(a^2 + b^2)$

which being independent of θ is always the same.

EXAMPLE 8 *Show that the moment of inertia of an elliptic area of mass M and equation,*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

about a diameter parallel to the axis of x is $-\frac{aM\Delta}{4(ab-h^2)^2}$,

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

Solution Equation of ellipse is $ax^2 + 3hxy + by^2 + 2gx + 2fy + c = 0$

Now transferring the origin to the centre of the ellipse, the equation of the ellipse becomes

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{ab - h^2} = 0 \qquad (1)$$

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$ (by geometry)

Putting $y = 0$, we get $ax^2 = -\frac{\Delta}{ab - h^2}$

Hence, if r is the semi-diameter parallel to the axis of x

$$r^2 = -\frac{\Delta}{a(ab - h^2)}$$

Equation (1) can be written as

$$-\frac{a}{c'}x^2 - 2\frac{h}{c'}xy - \frac{b}{c'}y^2 = 1 \qquad \text{where } c' = \frac{\Delta}{ab - h^2}.$$

(putting in the form $ax^2 + 2hxy + by^2 = 1$)

If α, β are the semi-axes of ellipse, then α^2, β^2 are the values of R^2 in the equation

$$\left(-\frac{a}{c'} - \frac{1}{R^2}\right)\left(-\frac{b}{c'} - \frac{1}{R^2}\right) = \left(\frac{h}{c'}\right)^2 \qquad \text{(by Coordinate Geometry)}$$

i.e.,
$$\frac{1}{R^4} + \frac{1}{R^2}\left(\frac{a}{c'} + \frac{b}{c'}\right) + \frac{ab - h^2}{c'^2} = 0.$$

$$\therefore \quad \frac{1}{\alpha^2\beta^2} = \frac{ab - h^2}{c'^2} = \frac{(ab - h^2)^3}{\Delta^2} \text{ putting value of } c'.$$

Hence, the moment of inertia about the diameter is

$$\frac{M}{4} \cdot \frac{\alpha^2\beta^2}{r^2} = -\frac{M}{4} \cdot \frac{\Delta^2}{(ab - h^2)^3} \cdot \frac{a(ab - h^3)}{\Delta}$$

$$= -\frac{aM\Delta}{4(ab - h^2)^2}$$

EXAMPLE 9 *Show that for a thin hemispherical shell of mass M and radius a, the moment of inertia about any line through the vertex is $\frac{2}{3}Ma^2$.*

Solution Let O be the vertex of the hemispherical shell, take the symmetrical radius OX and other two perpendicular lines OY and OZ as axes of reference.

The hemispherical shell is generated by the revolution of a quadrant of the circle about OX.

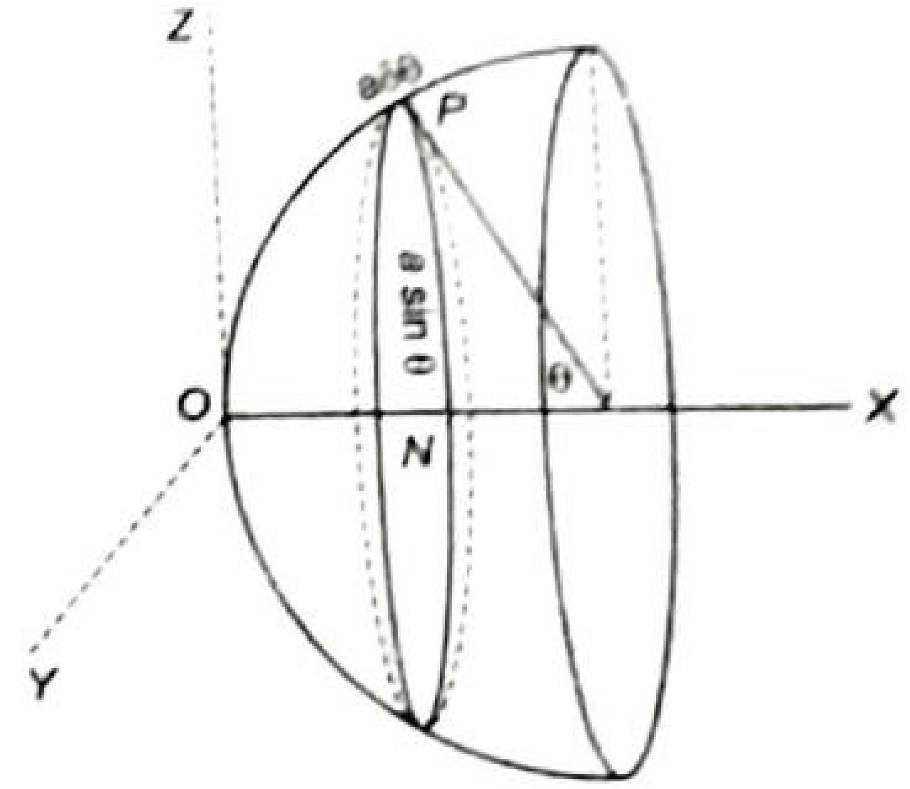

Fig.

$$A = \text{M.I. about } OX. = \int_0^{\pi/2} \rho\, 2\pi(a\sin\theta)\, a\, d\theta \cdot a^2\sin^2\theta$$

$$= 2\pi\rho a^4 \int_0^{\pi/2} \sin^3\theta\, d\theta = 2\pi\rho a^4 \cdot \frac{2}{3}$$

$$= \frac{4\pi\rho a^4}{3} = \frac{2}{3}Ma^2.$$

If B and C are moments of inertia about OY and OZ, then

$$B = C = \int_0^{\pi/2} \rho \cdot 2\pi a \sin\theta\, a\, d\theta \cdot \left[\frac{a^2\sin^2\theta}{2} + (a - a\cos\theta)^2\right]$$

$$= \pi\rho a^4 \int_0^{\frac{1}{2}\pi} \sin\theta\,(3 - 4\cos\theta + \cos^2\theta)\, d\theta$$

$$= \pi\rho a^4 \left[-3\cos\theta - 2\sin^2\theta - \tfrac{1}{3}\cos^3\theta\right]_0^{\pi/2}$$

$$= \pi\rho a^4 \left[3 - 2 + \frac{1}{3}\right] = \frac{4\pi a^4\rho}{3} = \frac{2}{3}Ma^2.$$

Also if D, E, F are the products of inertia about the axes, then $D = E = F = 0$, as coordinates of C.G. are $\left(\frac{a}{2}, 0, 0\right)$.

Now let $[l, m, n]$ be the direction-cosines of a line through O, then

M.I. about this line $= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Eln - 2Flm$

$$= \tfrac{2}{3}Ma^2(l^2 + m^2 + n^2) = \tfrac{2}{3}Ma^2.$$

EXAMPLE 10 *Show that the moment of inertia of an ellipsoid of mass M and semi axes a, b, c with regard to a diametral plane whose direction-cosines referred to principal planes are (l, m, n) is*

$$= \frac{1}{5}M(a^2l^2 + b^2m^2 + c^2n^2).$$

which represents an ellipsoid since A, B, C are all essentially positive. This is called the *momental ellipsoid* of the body at the point Q.

Since moment of inertia is essentially a positive quantity, being sum of a number of squares, it is clear that every radius vector r must be real.

Remark We know from Solid Geometry that for every ellipsoid there exist three mutually perpendicular diameters such that, if they be taken as axes of reference, the transformed equation has no terms involving yz, zx and xy.

These new axes of coordination are called the **Principal Axes** of the ellipsoid.

Let, referred to principal axes,the equation of the momental ellipsoid (1) be

$$A_1x^2 + B_1y^2 + C_1z^2 = MK^4 \quad \cdots(1)$$

The products of inertia about these new axes are clearly zero, because otherwise there would enter terms involving xy, yz or zx.

Hence, *for every body there exists at every point O, a set of three mutually perpendicular axes, which are three principal diameters of the momental ellipsoid at O, such that the products of inertia of the body about them, taken two at a time vanish.*

Cor. If the three principal moments at any point O are equal to each other, ellipsoid becomes a sphere. In this case every diameter is a principal diameter and all radii vectors are equal.

Also every straight line through O becomes principal axis at O and moments of inertia about them all are equal.

As an example–the perpendiculars from the centre of gravity of a cube on three coterminous faces are principal axes, because referred to them as axes,

$$\Sigma mxy = 0, \ \Sigma myz = 0, \ \Sigma mzx = 0$$

Moreover, three moments of inertia about them are equal, each being $\frac{2\,Ma^2}{3}$,

where $2a$ is the side of the cube.

Hence, moment of inertia about any line whose direction cosines are $[l, m, n]$ through the centre of the cube is

$$= \frac{2Ma^2}{3} l^2 + \frac{2Ma^2}{3} m^2 + \frac{2Ma^2}{3} n^2 = \frac{2Ma^2}{3}(l^2 + m^2 + n^2) = \frac{2Ma^2}{3}.$$

which is always the same.

1.10 MOMENTAL ELLIPSE

In the case of a plane lamina, if A, B are moments of inertia about the axes and F be the product of inertia about them, then moment of inertia of famina about a line OQ making an angle θ with OX, is given by

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta$$

Again if the point Q be such that this moment of inertia is inversely proportional to OQ^2, then

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta = \frac{MK^4}{OQ^2} = \frac{MK^4}{r^2}$$

which gives $\quad Ar^2\cos^2\theta - 2Fr^2\sin\theta\cos\theta + Br^2\sin^2\theta = MK^4$.

Thus, locus of Q in the cartesion form becomes

$$Ax^2 - 2Fxy + By^2 = MK^4$$

which represents an ellipse, (since A and B are essentially positive) and is called momental ellipse at the point O.

Note. The momental ellipse is the section of the momental ellipsoid at O by the plane of the lamina.

SOLVED EXAMPLES

EXAMPLE 1 *Show that the momental ellipsoid at the centre of an elliptic plate is* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2\left[\frac{1}{a^2} + \frac{1}{b^2}\right] = constant.$

Solution Take the major axis and minor axis of the ellipses and a perpendicular line OZ as the axes of reference. Then

$$A = \text{moment of inertia about } OX = \frac{1}{4} Mb^2$$

$$B = \text{moment of inertia about } OY = \frac{1}{4} Ma^2$$

$$C = \text{moment of inertia about } OZ = \frac{1}{4} M(a^2 + b^2).$$

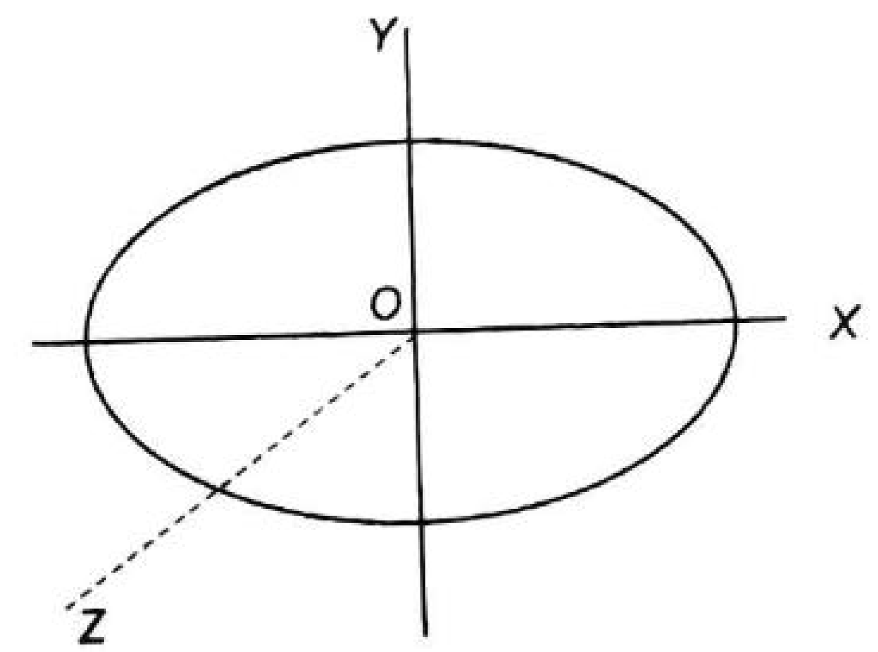

Fig.

If D, E and F are products of inertia about the axes y, z and z, x and x, y respectively we obviously have

$$D = E = F = 0.$$

Hence, the equation of the momental ellipsoid at O is

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$$

i.e.
$$\frac{1}{4}Mb^2x^2 + \frac{1}{4}Ma^2y^2 + \frac{1}{4}M(a^2+b^2)z^2 = \text{constant}.$$

Dividing by $\frac{1}{4}Ma^2b^2$, the equation of the ellipsoid at O becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) = \text{constant}.$$

EXAMPLE 2 *Show that the momental ellipsoid at the centre of an ellipsoid is $(b^2+c^2)x^2 + (a^2+c^2)y^2 + (a^2+b^2)z^2 = constant.$*

Solution Referred to principal axes as the axes of reference, the equation of the ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$A = \text{M.I. about } OX = \frac{1}{5}M(b^2+c^2),$$

$$B = \text{M.I. abut } OY = \frac{1}{5}M(c^2+a^2),$$

and $C = \text{M.I. about } OZ = \frac{1}{5}M(a^2+b^2),$

Products of Inertia, $D = E = F = 0.$

Hence, equation of the momental ellipsoid at the centre O is

$$\frac{1}{5}M[(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2] = \text{constant}$$

or
$$(b^2+c^2)x^2 + (c^2+a^2)y^2 + (a^2+b^2)z^2 = \text{constant}.$$

EXAMPLE 3 *Show that the equation of the momental ellipsoid at the corner of a cube of side 2a referred to its principal axes is*

$$2x^2 + 11(y^2+z^2) = c,$$

where c is constant.

Solution Let O be the corner of the cube and G its centre of gravity. We have to determine the equation of the normal ellipsoid at the centre O.

We know that moment of inertia of the cube (each side $= 2a$) about any axis through G.

$$= \frac{2}{3}Ma^2$$

Take line OG as the axis of x and two mutually perpendicular lines OY, OZ through O as the axes of y and z.

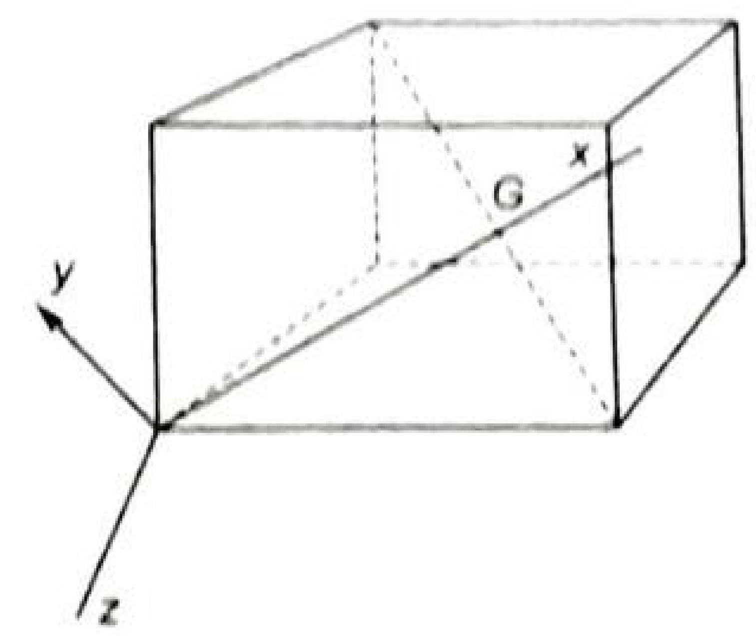

Fig.

As the product of inertia of the cube about any two mutually perpendicular lines through G is zero and coordinates of G referred to OX, OY, OZ, as areas are $(a\sqrt{3}, 0, 0)$ the products of inertia about the axes OX, OY, OZ, take in pairs are also zero. Therefore, OX, OY, OZ are the principal axes of the momental ellipsoid at O.

Now $\quad A = \text{M.I. about } OX = \frac{2}{3}Ma^2$

$B = \text{M.I. about } OY = \text{M.I. about a line through } G \parallel \text{ to } OY$
$+ \text{M.I. of mass } M \text{ placed at } G \text{ about } OY$

$$= \frac{2Ma^2}{3} + 3a^2M \qquad \text{as } OG = a\sqrt{3}$$

$$= \frac{11Ma^2}{3}$$

Similarly, $C = \text{M.I. about } OZ = \frac{11Ma^2}{3}$

Also, $\quad D = E = F = 0.$

Hence, equation of the momental ellipsoid at O is

$$\frac{2Ma^2}{3}x^2 + \frac{11Ma^2}{3}y^2 + \frac{11Ma^2}{3}z^2 = \text{constant}$$

or
$$2x^2 + 11(y^2+z^2) = c \text{ (constant)}$$

EXAMPLE 4 *Show that the momental ellipsoid at a point on the rim of a hemisphere is $2x^2 + 7(y^2+z^2) - \frac{15}{4}xz = constant.$*

Solution Let O be the point on the rim of the hemisphere, at which equation of the momental ellipsoid is to be determined.

Let the diameter through O be taken as x-axis, a line in the plane of the base and perpendicular to OX be taken as y-axis and a line $\perp$ to the plane of the base as z-axis.

$\therefore$ Coordinates of the centre of gravity G are $\left(0, 0, \frac{3a}{8}\right)$.

Let A, B, C be the moments and D, E, F the products of inertia about these axes.

Consider an elementary disc at a distance ξ from OX. Then radius of the disc $= \sqrt{(a^2-\xi^2)}$.

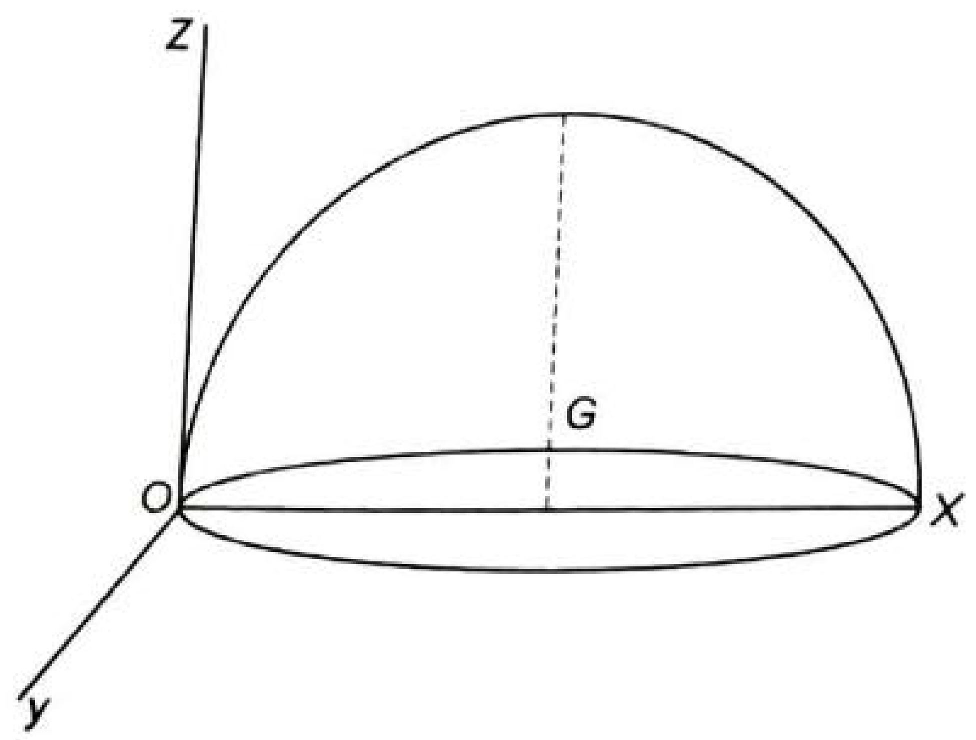

Fig.

M.I. of the disc about $OX = \pi(a^2-\xi^2)\, d\xi\rho\left[\frac{1}{4}(a^2-\xi^2)+\xi^2\right]$

$$\therefore \quad A = \frac{1}{4}\pi\rho\int_0^a (a^2-\xi^2)(a^2+3\xi^2)\,d\xi = \frac{1}{4}\pi\rho\int_0^a (a^4+2a^2\xi^2-3\xi^4)\,d\xi$$

$$= \frac{1}{4}\pi\rho\left[a^5+\frac{2}{3}a^5-\frac{3}{5}a^5\right] = \frac{4\pi\rho a^5}{15}$$

$$= \frac{2}{5}Ma^2 \text{ as } M = \frac{2}{4}\pi a^3\rho.$$

$$B = \frac{2}{5}Ma^2 + Ma^2 = \frac{7}{5}Ma^2$$

and $$C = \frac{7}{5}Ma^2.$$

Also $$D = M\cdot 0\times\frac{3}{8}a = 0,$$

$$E = M\frac{3}{8}a\times a = M\cdot\frac{3}{8}a^2$$

and $$E = M\cdot 0\times a = 0.$$

Hence, the equations of the momental ellipsoid at O is

$$Ax^2+By^2+Cz^2-2Dyz-2Ezx-Fxy = \text{constant}$$

i.e., $$Ma^2\left[\frac{2}{5}x^2+\frac{7}{5}y^2+\frac{7}{5}z^2-\frac{3}{4}xz\right] = \text{constant}$$

or $$2x^2+7(y^2+z^2)-\frac{15}{4}xz = \text{constant}.$$

EXAMPLE 5 *Show that the momental ellipsoid at a point on the edge of the circular base of a thin hemispherical shell is*

$$2x^2+5(y^2+z^2)-3zx = \text{constant}.$$

Solution Let a be the radius of the shell. Let O be the point on the circular edge at which momental ellipsoid is to be determined.

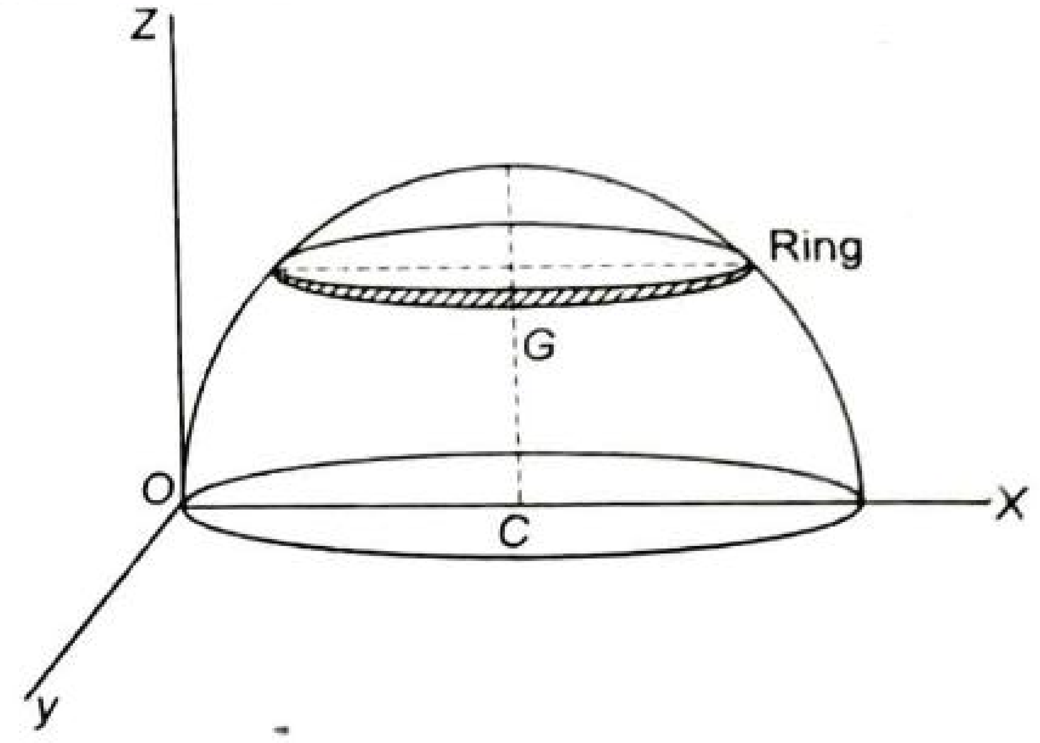

Fig.

$$A = \text{moment of inertia about } OX = \frac{2}{3}Ma^2.$$

B = M.I. about OY

= M.I. about a parallel line through $C + Ma^2$

$$= \frac{2}{3}Ma^2 + Ma^2 = \frac{5}{3}Ma,$$

Also $$C = \frac{5}{3}Ma^2$$

Coordinates of centre of gravity are $\left(1, 0, \frac{1}{2}a\right)$

$$\therefore \quad D = F = 0, \quad E = Ma\cdot\frac{1}{2}a = \frac{1}{2}Ma^2$$

Hence, equation of the momental ellipsoid at O is

$$\frac{2}{5}Ma^2x^2+\frac{5}{3}Ma^2y^2+\frac{5}{3}Ma^2z^2-2\cdot\frac{1}{2}Ma^2zx = \text{constant}$$

i.e., $$2x^2+5(y^2+z^2)-3zx = \text{constant}.$$

EXAMPLE 6 *Prove that the equation of the momental ellipsoid at a point on the circular edge of a solid cone is*

$$(3a^2+2h^2)x^2+(23a^2+2h^2)y^2+26a^2z^2-10ahxz = \text{constant}$$

where h is the height and a the radius of the base.

Solution Let O be a point on the circular edge of the cone, at which we want to determine the momental ellipsoid.

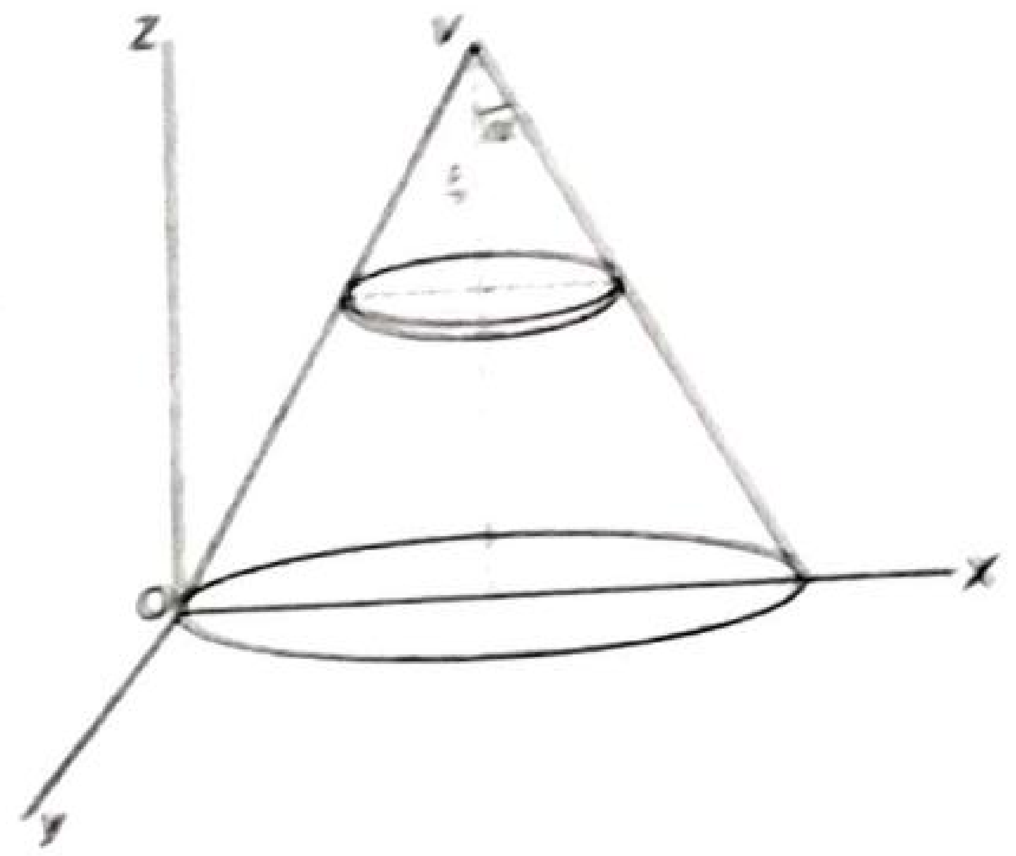

Fig.

Consider a disc of breadth $\delta\xi$ at a depth ξ from the vertex V of the cone.

A = moment of inertia about OX

$$= \int_0^h \pi\xi^2 \tan^2\alpha \cdot \rho\left[\frac{\xi^2\tan^2\alpha}{4} + (h-\xi)^2\right] d\xi$$

$$= \pi\tan^2\alpha \cdot \rho \int_0^h \left[\frac{\xi^4\tan^2\alpha}{4} + h^2\xi^2 - 2h\xi^3 + \xi^4\right] d\xi$$

$$= \pi\tan^2\alpha \cdot \rho\left[\frac{\xi^5\tan^2\alpha}{20} + \frac{h^2\xi^3}{3} - \frac{h\xi^4}{2} + \frac{\xi^5}{5}\right]_0^h$$

$$= \pi\tan^2\alpha \cdot \rho h^5\left[\frac{\tan^2\alpha}{20} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right]$$

$$= \pi\tan^2\alpha \cdot \rho h^5\left[\frac{\tan^2\alpha}{20} + \frac{1}{30}\right]$$

But $M = \frac{1}{3}\pi h^3 \tan^2\alpha \cdot \rho$ and $\tan\alpha = \frac{a}{h}$

$$= 3\,Mh^2\left[\frac{a^2}{20h^2} + \frac{1}{30}\right]$$

$$= \frac{M}{20}(3a^2 + 2h^2).$$

B = moment of inertia about OY

= moment of inertia about a parallel axis through centre + Ma^2

$$= \frac{M}{20}(3a^2 + 2h^2) + Ma^2 = \frac{M}{20}(23a^2 + 2h^2)$$

C = moment of inertia about parallel line through centre + Ma^2

$$= Ma^2 + \int_0^h \pi\xi^2 \tan^2\alpha \cdot \rho\frac{\xi^2\tan^2\alpha}{2} d\xi$$

$$= Ma^2 + \pi\tan^4\alpha \cdot \rho\frac{h^5}{10}$$

$$= Ma^2 + \frac{3M}{10}a^2 = \frac{13\,Ma^2}{10}$$

There is symmetry about axis of y and coordinates of the centre of gravity G are $\left(a, 0, \frac{1}{4}h\right)$.

$$\therefore \quad D = F = 0$$

and

$$E = Ma \cdot \frac{1}{4}h = \frac{1}{4}Mah$$

Hence, substituting different values in

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant},$$

the equation of momental ellipsoid at O is

$$\frac{M}{20}(3a^2 + 2h^2)x^2 + \frac{M}{20}(23a^2 + 2h^2)y^2 + \frac{13}{10}Ma^2z^2 - 2\frac{1}{4}Mahxz = \text{constant}$$

or

$$(3a^2 + 2h^2)x^2 + (23a^2 + 2h^2)y^2 + 26a^2z^2 - 10ahxz = \text{constant}.$$

EXAMPLE 7 *Find the momental ellipsoid at any point O of a material straight rod AB of mass M and length $2a$.*

Solution Let G be the centre of gravity of the uniform rod and O the point at which equation of the momental ellipsoid is to be determined. Let $OG = c$.

A = moment of inertia about $OX = 0$,

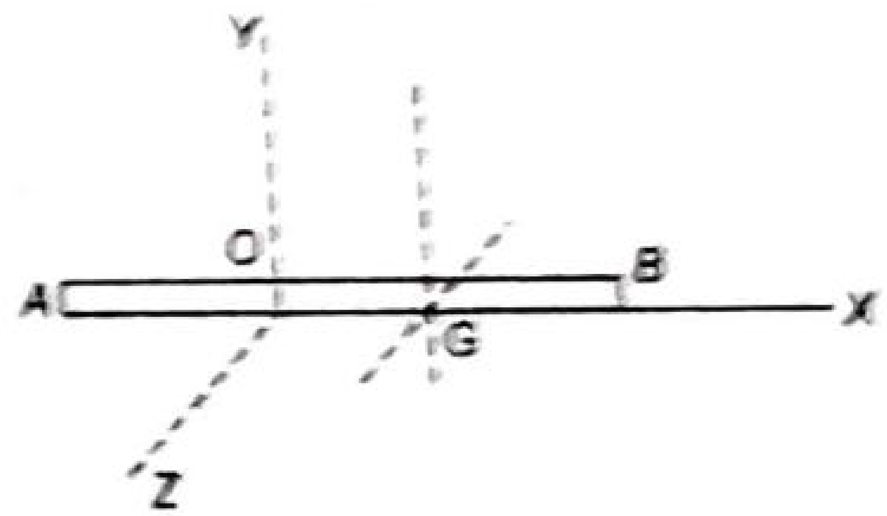

Fig.

B = moment of inertia about OY

= M.I. about a parallel line (*i.e.*, a through $G \perp$ to the rod) + $M \cdot OG^2$

where $OG = c$